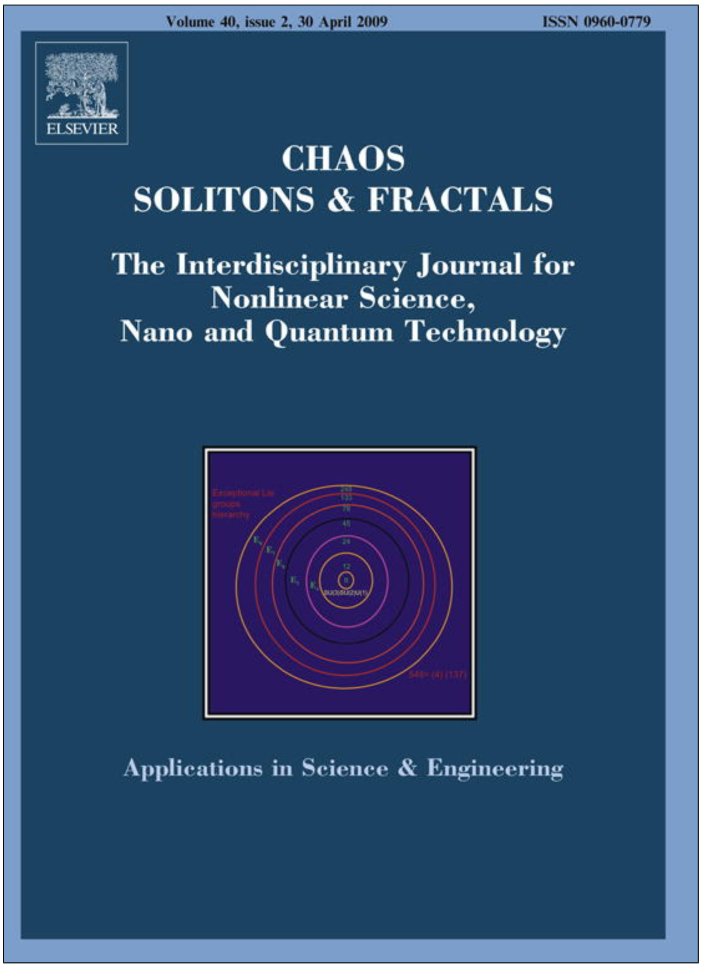
Volume 40, issue 2, 30 April 2009
ISSN 0960-0779
ELSEVIER
CHAOS
SOLITONS & FRACTALS
The Interdisciplinary Journal for
Nonlinear Science,
Nano and Quantum Technology
Applications in Science & Engineering

Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals 40 (2009) 717–730
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Nonlinear dynamics in a business-cycle model with logistic population growth

Serena Brianzoni, Cristiana Mammana, Elisabetta Michetti *

Department of Economic and Financial Institutions, University of Macerata, Italy

Accepted 13 August 2007

Abstract

We consider a discrete-time growth model of the Solow type where workers and shareholders have different but constant saving rates and the population growth dynamics is described by the logistic equation able to exhibit complicated dynamics. We show conditions for the resulting system having a compact global attractor and we describe its structure. We also perform a mainly numerical analysis using the critical lines method able to describe the strange attractor and the absorbing area, in order to show how cyclical or complex fluctuations may be produced in a business-cycle model. We study the dynamic behaviour of the model under different ranges of the main parameters, i.e. the elasticity of substitution between the two production factors and the one in the logistic equation (namely μ). We prove the existence of complex dynamics when the elasticity of substitution between production factors drops below one (so that capital income declines) or μ increases (so that the amplitude of movements in the population growth rate increases).

1. Introduction

In recent years an increasing interest in the study of nonlinear dynamical systems in economics has been observed as complex dynamics often emerge in economic models. A large number of contributions to the study of discrete and continuous dynamical systems and their applications to many fields of economics has been produced (see, for instance [20]) and many special issues containing applications in economics and finance of dynamical system theory have been published (see, among others, the special issue Complex Dynamics in Economic and Social Systems 1996, or, more recently, the special issue Dynamic Modelling in Economics and Finance in honour of Professor Carl Chiarella, 2006, both published in Chaos, Solitons & Fractals).

Among other fields in economics, nonlinear dynamical systems emerge quite naturally in the description of the economic growth or business cycles in a country, so many authors faced with the question of the asymptotic behaviour of the economic growth model, its properties and, hence, the emergence of chaos (see, for instance [32,15,40]).

Dynamic economic growth models have often considered the standard, one-sector neoclassical model by Ramsey [33] or the Solow–Swan model (see [36,37]). Both these dynamic models show that the system monotonically converges to the steady state (i.e. the capital per capita equilibrium) so neither cycles nor complex dynamics can be observed.

* Corresponding author. Tel.: +39 0733 2583243; fax: +39 0733 2583205.
E-mail address: michetti@unimc.it (E. Michetti).

0960-0779/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2007.08.041

However while Ramsey's assumption on savings behaviour corresponds to maximization of the infinite discounted sum of utility of a representative consumer who lives infinitely, in the Solow–Swan model constant average propensity to save is assumed.

In order to investigate the possibility of complex dynamics to be shown in optimal growth models, other authors have considered more sector models. For example, Benhabib and Nishimura [5] and Boldrin and Montrucchio [10] demonstrated that chaotic dynamics can be produced within two-sector models of optimal growth. Other examples of fluctuations in two-sector models are those by Becker and Tsyganov [4] and Nishimura and Yano [29]. In particular Nishimura and Yano considered the Leontief two-sector model by showing the possibility of optimal chaos. Also Ishiyama and Saiki [22] reached a similar result as they proved the existence of a chaotic attractor in a macroeconomic growth cycle model of two countries with different fiscal policies.

Differently, while considering the one-sector model, some authors (i.e. [23,24,31,34]) studied the question whether the different saving propensities of two groups (labor and capital) might influence the final dynamics of the system. The question of differential savings between groups of agents was originally posed within the Harrod–Domar model of fixed portion [21]. Obviously different but constant saving propensities make the aggregate saving propensity non-constant and dependent on income distribution so that multiple and unstable equilibria may occur. However, qualitative dynamics is still simple.

More recently, Szydlowski et al. [39] reconsidered the classical Kaldor model in which capital stock changes are caused by past investment decisions. Their study proves the existence of asymmetric (two periodic) cycles in the Kaldor model with time-to-build. Later, in Szydlowski and Krawiec [38] conditions for stability of limit cycles on the centre manifold are given.

Differently form previous contributions, our starting point is the discrete-time Solow–Swan growth model with differential saving rates between workers and shareholders as proposed by Bohm and Kaas [9]. The authors, as typical, assume that the labor force grows at a constant rate $n \geqslant 0$. This last hypothesis is usually assumed in standard economic growth theory. However, one implication of a constant population growth rate is that population grows exponentially, this assumption is unable to explain possible fluctuations in the growth rate. For this reason a number of economic growth model with endogenous population growth has been proposed (see, among others [30,41]).

Brianzoni et al. [11] and Cheban et al. [14] recently investigated the neoclassical growth model with labor force dynamics described by the Beverton–Holt equation (see [6]) while assuming CES production function. The authors proved that multiple equilibria are likely to emerge, and they provided conditions on the parameters. They also show that complex dynamics can be exhibited if the elasticity of substitution between production factors is low enough.

However, while in the abovementioned works the authors considered simple population dynamic laws such that the population growth rate converges to a unique globally stable fixed point, in this paper we formalize the population growth evolution with the well-known logistic map, able to exhibit more complex dynamics as cycles of every order or chaos.

In fact, as described by Maynard Smith [26], in many countries a more realistic population growth model would have the following properties: (1) when population is small in proportion to the environmental carrying capacity, then it grows at a positive constant rate and (2) when population is larger in proportion to the environmental carrying capacity, the resources become relatively more scarce and, as result, this must affect the population growth rate negatively. Since the logistic map satisfies both properties, we describe the population growth using the logistic function rather than the exponential one.

Verhulst (see [35]) was the first who proposed to model population growth with the logistic equation. Other authors made the same choice: Brida et al. [12] analyzed the neoclassical Solow model with growth of population described by a generalized logistic growth law, Accinelli and Brida [2] studied the Ramsey model of optimal economic growth with the logistic equation (Richard's law).

On the basis of our assumptions, the resulting model T is a bidimensional triangular dynamic system able to generate endogenous fluctuations for certain values of the parameters. We study the long run behaviour of the system according to the parameter values. More precisely we perform both a theoretical and a numerical analysis. In fact first we prove that T admits the compact global attractor for some values of the parameters and we describe its structure; second we show how the global dynamics of the model can be analyzed by the study of some global bifurcations that change the shape of the chaotic attractor as some parameters vary. These bifurcations are analyzed by using the critical curves method.[1] The results obtained aim at confirming that production function's elasticity of substitution plays a central role in the creation and propagation of complicated dynamics as in models with explicitly dynamic optimizing behaviour by

[1] Other works in this area are, for example, those by Bischi et al. [8], Bischi and Gardini [7] and Dieci et al. [17].

private agents.[2] More precisely, when the elasticity of substitution between production factors drops below one (so that capital income declines) then the dynamics becomes more and more complex, providing that the business-cycle model produces irregular fluctuations. Another parameter responsible for the creation of complicated dynamics is the one in the logistic equation, related to the amplitude of fluctuations in the population growth rate.

The paper is organized as follows. In Section 2, we present the triangular growth model with differential savings and logistic population growth. In Section 3, we perform a local analysis on the stability of the fixed and periodic points owned by the system. In Section 4, we prove that system T has the compact global attractor and we describe its structure. In Section 5, we use the critical lines method to describe the attractor of the system and to obtain the invariant absorbing area where asymptotic dynamics is confined. We also present numerical simulations. We conclude the paper in Section 6.

2. The model

The model we study is obtained while considering the standard, neoclassical one-sector growth model where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm and Kaas [9]. The one-dimensional map describing the evolution of the capital per capita k_t is defined as

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + s_w(F(k_t) - k_tF'(k_t)) + s_rk_tF'(k_t)]$$

where $\delta \in (0, 1)$ is the depreciation rate of capital, $s_w \in (0, 1)$ and $s_r \in (0, 1)$ are the constant saving rates for workers and shareholders, respectively, and F is the production function, while n is the constant population growth rate.

Differently from Bohm and Kaas [9] we consider that the production function F: $R_+ \to R_+$, mapping capital per worker k into output per worker y, is of the CES type, that is given by

$$y = F(k) = (1+k^\rho)^{\frac{1}{\rho}}. \tag{1}$$

Concerning the new ingredient of the study herewith proposed, we assume that the labor force growth rate is not constant, so that we take into account the possibility of fluctuations in the population growth rate. In such a way we distinguish from previous works (as, e.g. Brianzoni et al. [11]) where an iterative scheme having simple features has been proposed. In fact we describe the evolution of the population growth rate within the logistic map that satisfies some economic desired properties, that has been previously recalled.

More precisely, we study the following system describing capital per capita (k) and population growth rate (n) dynamics assuming that population growth rate is described by the well-known logistic map. Consequently system $T = (n', k')$ is given by

$$T := \begin{cases} n' = \mu n(1-n) \\ k' = \frac{1}{1+n}[(1-\delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}}(s_w + s_rk^\rho)] \end{cases} \tag{2}$$

where $\rho \in (-\infty, 1)$, $\rho \neq 0$ is a parameter related to the elasticity of substitution between labor and capital[3] and $\mu \in (1, 4)$ for the dynamics generated by the logistic map being economically interesting, i.e. not explosive.

We get a discrete-time dynamical system described by the iteration of the following triangular map of the plane:

$$T := \begin{cases} n' = f(n) \\ k' = g(n,k) \end{cases}, \tag{3}$$

where k is assumed to be positive and, moreover, for $\rho < 0$, $g(n, 0) = 0$, $\forall n \in R_+$, for g being continuous.

3. Stability of fixed and periodic points

The equilibrium points (or steady states) of map T are all the solutions of the algebraic system $T(n, k) = (n, k)$. The first equation says that the fixed points belong to the lines $n = 0$ and $n = n^*$, where $n^* = \frac{\mu-1}{\mu}$. From the second equation we have that the k-values are the fixed points of the one-dimensional maps $g_0(k) := g(0, k)$ and $g_{n^*}(k) := g(n^*, k)$ that have been largely studied in Brianzoni et al. [11]. In that work it has been proved that maps g_0 and g_{n^*} have up to three fixed points depending on the parameter values of the model. As a consequence multiple equilibria are likely to emerge

[2] The survey paper by Becker [3] covers many of these models.
[3] Remember that the elasticity of substitution between the two production factors is given by $\frac{1}{1-\rho}$.

in our model differently from the result in Bohm and Kaas [9], where a production function satisfying weak Inada conditions has been considered.

For the local stability analysis of the fixed points, denote with $J(n, k)$ the Jacobian matrix of system T and recall the following property.

Property 3.1. *The eigenvalues of* $J(n, k)$ *are always real, given by* $\lambda_1 = f'(n)$ *and* $\lambda_2 = \frac{\partial g}{\partial k}(n,k)$. *Any fixed point of T is therefore either a node or a saddle. The same is true for any cycle* $O_m = \{(n_i, k_i), i = 1, 2, \ldots, m\}$ *of T whose eigenvalues are given by* $\lambda_1 = \prod_{i=1}^{m} f'(n_i)$ *and* $\lambda_2 = \prod_{i=1}^{m} \frac{\partial g}{\partial k}(n_i, k_i)$.

The local stability analysis of a fixed point can be carried out by studying the localization of the eigenvalues of the Jacobian matrix in the complex plane, and it is well known that a sufficient condition for the local stability is that both the eigenvalues are inside the unit circle in the complex plane. The triangular structure of system T simplifies our analysis, since, according to Property 3.1 the Jacobian matrix of T has real eigenvalues located on the main diagonal, given by

$$\lambda_1(n) = \mu - 2\mu n, \qquad \lambda_2(n,k) = \frac{\partial g}{\partial k}(n,k).$$

Trivially, $\lambda_1(0) > 1$ so that fixed points belonging to the line $n = 0$ can be both saddle points or unstable nodes, while $|\lambda_1(n^*)| < 1$ if and only if $\mu \in (1, 3)$, so that in this last case the line $n = n^*$ attracts trajectories starting from initial conditions (n_0, k_0), s.t. $n_0 \in (0, 1)$.

Consider now $\lambda_2(n, k)$. In Brianzoni et al. [11] it has been proved that $g(n, k)$ can be strictly monotonic or bimodal in k and sufficient conditions has been pursued. We brief recall such results.

Property 3.2. *Consider system T.*

(*i*) *If* $\rho \in (0, 1)$ *and* $s_r < \delta$ *then* g_0 *and* $g_{n^\star}$ *have a unique positive globally stable fixed point*;
(*ii*) *If* $\rho < 0$ *and* $s_r < s_w$ *then* g_0 *and* $g_{n^\star}$ *have up to three fixed points, the unstable one separates the basin of attraction of the stable ones*;
(*iii*) *A* $\bar{\rho} < 0$ *does exist such that if* $\bar{\rho} < \rho < 0$ *and* $s_r > s_w$ *then* g_0 *and* $g_{n^\star}$ *have a unique positive globally stable fixed point.*

The previous property refers to the cases in which $g(n, k)$ is strictly monotonic in k, differently, in the open case, i.e. ρ low enough ($\rho < \bar{\rho} < 0$) and $s_r > s_w$, the map $g(n, k)$ is bimodal in k.

Let us recall some general results for the logistic map f (see [27,16]). If $1 < \mu < \mu_1 := 3$, n^* is an attracting fixed point. Let $\mu_\infty \approx 3, 569\ldots$ being the Feigenbaum number.[4] If $\mu_k < \mu < \mu_{k+1} < \mu_\infty$, $(k = 1, 2, 3, \ldots)$, then f has an attracting cycle with period $m = 2^k$. Map (f, [0, 1]) in this case has an asymptotically stable m-cycle (i.e. it attracts any i.c. belonging to (0, 1)), namely $O_m = \{n_1, n_2, \ldots, n_m\}$. Consequently, for all $\mu \in (1, \mu_\infty)$ the structure of the attractor I_μ of f is sufficiently simple: there exists a unique attracting m-periodic point which attracts all trajectories from (0, 1).

We first consider parameter values such that Property 3.2 holds. As a matter of facts, in all the cases abovementioned, if $\mu \in (1, 3)$ fixed points of T can be both stable nodes or saddle points, but the attractor of T is quite simple since it consists of a countable set of points. No complicated dynamics can be produced. The attractor still has a simple structure if $\mu \in [3, \mu_\infty)$, see Fig. 1, panel a where the attractor of T is a four period cycle that belongs to the lines $n = n_0$ and $n = n_1$, where n_0 and n_1 are the two periodic points of the attracting 2-period cycle of f. The parameters have been chosen such that property (iii) of 2 is fulfilled.

The resulting economic consideration is that if the elasticity of substitution between production factors is greater than one ($\rho > 0$) and μ is low enough ($\mu < 3$) no complicated dynamics is exhibited; in fact the capital income monotonicity property holds.[5] Observe also that as $\rho \to 0^+$, the CES production function approaches the Cobb-Douglas form and the elasticity of substitution between the two factors is equal to 1. On the other hand, if μ increases, i.e. $\mu \in (3, \mu_\infty)$, fluctuations are exhibited by our model: it is due to cycles in the population growth rate.

If $\rho < 0$ the elasticity of substitution between production factors is lesser than one and the capital income monotonicity property may fail to hold. It is just in such a case where the two propensities to save play an important role. In fact if workers save more than shareholders, the capital income monotonicity property still holds, while if $s_r > s_w$ then ρ must be pushed sufficiently down for capital income monotonicity property not being verified and fluctuations to be observed. The following proposition comes directly from the previous analysis.

[4] It is a universal constant for functions approaching chaos via period-doubling, see [18].

[5] Remember that the capital income is given by $kF'(k)$. It is easy to verify that if $\rho \in (0, 1)$ the capital income is strictly monotonic in capital.

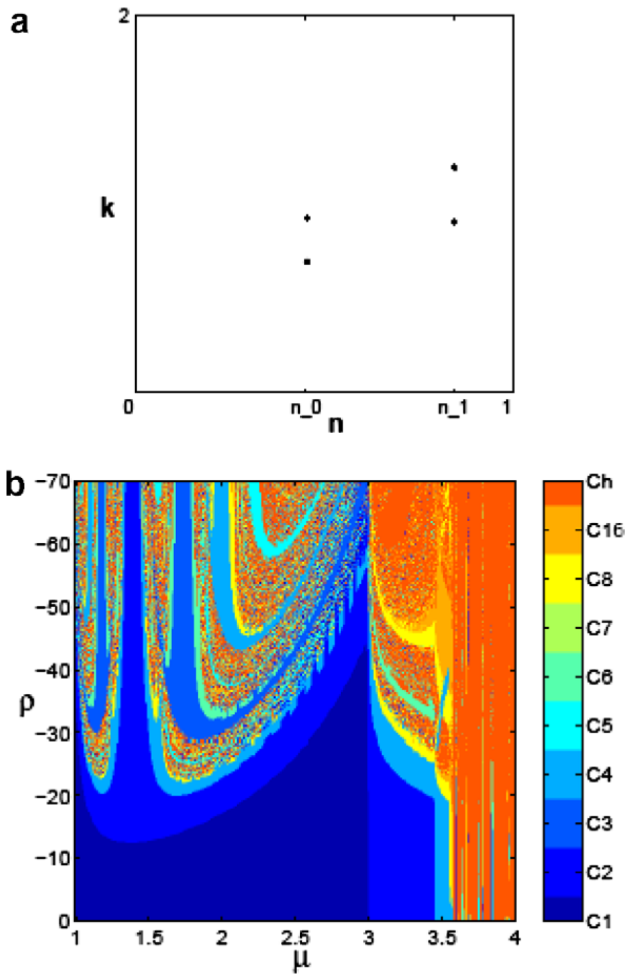

Fig. 1. (a) Attractor for $\rho = -50$, $\delta = 0.01$, $s_w = 0.1$, $s_r = 0.9$ with $\mu = 3.4$ and an i.c. $n_0 \in (0, 1)$ with $k_0 \neq 0$. The logistic map has a unique cycle-2 asymptotically stable whose points are $n_0 \simeq 0.45$ and $n_1 \simeq 0.84$. Attractor of T belongs to two different lines having equations $n = n_0$ and $n = n_1$ ad it is a 4-period cycle. (b) Cycle cartogram in (μ, ρ) plane for $\delta = 1.5$ and the other parameters as in panel a.

Proposition 3.3. *Assume the same assumptions of property (2) and $\mu < \mu_\infty$. Then the attractor of system T consists either of fixed or periodic points.*

As a consequence, we now focus on the case with $\mu > \mu_\infty$ and/or $s_r > s_w$ such that $\rho < 0$ is low enough. For instance a fixed point loses stability if μ increases (according to the well-known dynamic properties of the logistic map) or if ρ is lesser than a given value $\bar{\rho}$, depending on the other parameters of the model. Observe that the economic dynamics becomes more and more complicated as the elasticity of substitution between production factors is low enough (Leontief technology is just the extreme example of this case).

Fig. 1, panel b shows the two-dimensional bifurcation diagram in the plane (μ, ρ) for the case $s_r > s_w$. It contains a cycle cartogram showing a two-parametric bifurcation diagram qualitatively. Each color represents a long-run dynamic behaviour for a given point on the parameter plane and for a generic initial condition. Observe that if ρ is great enough and $\mu < \mu_\infty$ the final dynamics is quite simple while it becomes more and more complex as ρ decreases or μ increases, suggesting that greater values of μ increase the complexity of the asymptotic dynamical behaviour of the system, as well as a ρ value small enough.

4. Global dynamics

In this section we study the global dynamics of system (T, R^2_+). In particular, we first prove a general result stating conditions on the parameters for the existence of the compact global attractor and, then, we describe its structure.

Recall the following definition of the global attractor.

Definition 4.1. A nonempty compact set $C \subset R^2_+$ is the global attractor of the dynamical system (T, R^2_+) if the following conditions are fulfilled:

(a) C is invariant with respect to (T, R^2_+);
(b) C attracts all the bounded subsets from R^2_+.[6]

Let us give conditions for the existence of the compact global attractor.

Proposition 4.2. *For all* $\{\rho \in (0,1)\} \cap \{s_r < \delta\}$ *or* $\rho < 0$, *the dynamic system* (T, R^2_+) *admits the compact global attractor* $\Lambda \subset [0,1] \times [0, M]$, *where* M *is a positive real number.*

Proof. Notice that the map f acts from interval [0, 1] into itself, in other words [0, 1] is a trapping region, i.e. a closed region positively invariant.[7] Consequently, f admits a compact global attractor $J \subseteq [0, 1]$.

Assume first $\rho \in (0,1)$ and $s_r < \delta$, then a straightforward calculation shows that $\forall \varepsilon > 0$, $\exists\ L > 0$ such that $g(n, k) < (1 - \delta + s_r + \varepsilon)k$, $\forall k > L$. In order to prove that the trajectory starting from a point (n_0, k_0) at least one time intersects the set $[0,1] \times [0, L]$, we suppose that this statement is false. Then there exists a point $(n_0, k_0) \notin [0,1] \times [0, L]$ such that $T^t(n_0, k_0) \notin [0,1] \times [0, L]$, $\forall t \in Z_+$. Taking into account the previous considerations on f it must be $k_t = g^t(n_0, k_0) > L\ \forall t \in Z_+$. Nevertheless, assuming $\varepsilon < \delta - s_r$, we have

$$k_t < (1 - \delta + s_r + \varepsilon)^t k_0 \to 0 \quad \text{as } t \to +\infty \tag{4}$$

so that we obtain a contradiction.

Consider now $k_0 < L$. If there exists $t_0 \in Z_+$ such that $k_{t_0} > L$, where $t_0 = \min\{t \in Z_+ : k_t > L\}$, then from Eq. (4) we obtain $k_t < (1 - \delta + s_r + \varepsilon)^{t - t_0} k_{t_0}$ for $t \in Z_+$ such that $k_t > L$, which proves that $\{k_t\}_{t \in Z_+}$ is bounded under the assumption $\varepsilon < \delta - s_r$, i.e. $k_t \leqslant L'$, where $L' \geqslant L$ is a positive real number. Consequently, the set $[0,1] \times [0, L']$ is attracting and positively invariant.

Second, let $\rho < 0$ and the map $g(n,k) = \frac{1}{1+n}[(1-\delta)k + j(k)]$ where $j(k) := (k^\rho + 1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)$. Being $\lim_{k \to +\infty} j(k) = s_w$, then $\exists N > 0$ such that $g(n,k) < [(1-\delta)k + N]\ \forall k \geqslant 0$. Consequently,

$$k_t = g^t(n_0, k_0) < (1-\delta)^t k_0 + (N) \sum_{i=0}^{t-1} (1-\delta)^t \to \frac{N}{1-\delta} \quad \text{as } t \to +\infty, \tag{5}$$

$\forall k_0 \geqslant 0$. As a consequence the trajectory starting from a point (n_0, k_0) at least one time intersects the set $[0,1] \times [0, N]$ and never leaves it.

Finally, define $M = L'$ if $\rho \in (0,1)$ and $M = N$ if $\rho < 0$. Since $[0,1] \times [0, M]$ is a compact, positively invariant and attracting set for T, then $\Lambda = \cap_{t \geqslant 0} T^t([0,1] \times [0, M])$ is a compact invariant set which attracts $[0,1] \times [0, M]$, i.e. $\forall \varepsilon > 0 \exists t_\varepsilon$ such that $T^t([0,1] \times [0, M]) \in B(\Lambda, \varepsilon)\ \forall t > t_\varepsilon$, where $B(\Lambda, \varepsilon)$ is a ε-neighborhood of Λ. □

In order to investigate the structure of the attractor Λ, we consider the case in which hypotheses of Proposition 4.2 are fulfilled. Furthermore, being $\Lambda \subset [0,1] \times [0, M]$, it is convenient to define the set $D := [0,1] \times [0, M]$ and consider the restriction of the system on D, i.e. the subsystem (T, D).

In particular, we first analyze the dynamics along the sets $k = 0$ and $n = 0$. This choice moves from the fact that, if $\rho < 0$, the two coordinates axes are both invariant sets, and that for $\rho \in (0,1)$ and $s_r < \delta$ the unique invariant axes is $n = 0$ while $k = 0$ is a repelling set.

About the first restriction $k = 0$, notice that the properties of the trajectories embedded in the invariant axis $k = 0$ can be easily deduced from the well-known properties of the standard logistic map. More specifically $n = 0$ is a fixed point and for $n_0 = 1$ the trajectory converges to zero at the second step, while every initial condition $n_0 \in (0,1)$ generates bounded trajectories which converge to a unique attractor included in a trapping interval.[8]

About the second restriction $n = 0$, we refer to the properties of map $g_0(k)$ that have been recalled in the previous section.

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[6] See [13].
[7] A set $E \subseteq R^2_+$ is positively (negatively) invariant if $T^t(E) \subseteq E (T^t(E) \supseteq E)$, $\forall t \in Z_+$. E is called invariant when it is both positively and negatively invariant.
[8] A trapping region is a closed region positively invariant.

Since we considered the dynamics of the system along the sets $k = 0$ and $n = 0$, and being $f(1) = 0$, in what follows we refer to the system (T, D'), where D' is defined as $D' := (0, 1) \times (0, M]$.

By taking into account the previous considerations on the invariant sets of T and also considering the arguments used to prove Proposition 4.2, it is easy to conclude as in the following proposition.

Proposition 4.3. *For any initial condition $(n_0, k_0) \in D'$, all the images $T^t(n_0, k_0)$ of any rank t belong to the set D'.*

Given the triangular structure of our system, the dynamics of T is closely related to the one of the one-dimensional map f, as stated in the following property (see [25,19] for a wider discussion).

Property 4.4. *If $O_m = \{n_1, n_2, \ldots, n_m\}$ is an $m-$ cycle of the map f, then the restriction of the map T^m to any vertical lines $n = n_i$, $i = 1, \ldots, m$, is trapping on that line. If the m-cycle of f is attracting (resp. repelling) then the vertical lines $n = n_i$, $i = 1, \ldots, m$, are attracting (resp. repelling) for T^m.*

From the previous property it is easy to see that any bifurcation of the one-dimensional map f gives a bifurcation of system T. In particular a fold bifurcation of f creates a couple of cyclical trapping lines of T (one repelling and one attracting). At a flip bifurcation of a cycle of f, trapping cyclical vertical lines from attracting (for T) become repelling and new cyclical attracting lines are created.

Taking into account Property 4.4 and the fact that f has an attractive m-period cycle for $\mu \in (1, \mu_\infty)$, it is easy to deduce the following proposition.

Proposition 4.5. *Let $\{\rho \in (0, 1)\} \cap \{s_r < \delta\}$ or $\rho < 0$. If $\mu \in (1, \mu_\infty)$, T admits m trapping cyclical attracting lines given by $J_\mu = \cup_{i=1}^{m}[n_i] \times (0, M]$.*

The results herewith obtained are quite general as they proved that if $\mu \in (1, \mu_\infty)$, the attractor of T belongs to the m attracting segment of J_μ. Anyway it may consist of fixed points, periodic cycles or a more complex set.

5. Complex dynamics and absorbing areas

The main purpose of this section is to analyze the properties of critical curves and to illustrate their applications to the study of the triangular growth model (T, D'). This method is important in practical problems because it can be used to define compact regions of the phase plane that acts as trapping bounded sets, inside which asymptotic trajectories are confined.

We first recall some properties about the trapping interval in which the asymptotic dynamics of the logistic map occurs.

For $\mu \in (1, 4)$, every initial condition $n_0 \in (0, 1)$ generates bounded trajectories which converge to a unique attractor included in the trapping interval $I = [c_1, c]$, where $c = \frac{\mu}{4}$ is the maximum value (critical point of rank-1) and $c_1 = f(c)$ is the critical point of rank-2. In particular for $\mu \in (1, 3)$ the trajectories converge to the fixed point n^*, while for $\mu \in (3, 4)$ the attractor can be a cycle of period m or a chaotic attractor (at $\mu = 3$ the first period doubbling bifurcation occurs). Notice that the chaotic attractor, as any attractor of the logistic map, is included in I. For $\mu = 4$ the image c_1 of the critical point c is mapped by f into the repelling fixed point $n = 0$. This represents the final bifurcation, after which a generic initial condition $n_0 \in (0, 1)$ generates a divergent sequence.

Since I is a trapping interval for the logistic map, it is possible to state the following proposition.

Proposition 5.1. *For all $\{\rho \in (0, 1)\} \cap \{s_r < \delta\}$ or $\rho < 0$, the set $A := I \times (0, M]$ is globally attracting and positively invariant for system (T, D').*

Proof. Taking into account Proposition 4.2, the proof follows immediately from the fact that the interval I is a trapping region for f in (0, 1). Moreover, remember that the set $k = 0$ is repelling if $\rho \in (0, 1)$ and invariant for $\rho < 0$, so that $I \times (0, M]$ is attracting and positively invariant for (T, D'). □

As a consequence of the previous proposition, the attractor Λ belongs to A and, denoting with $B(\Lambda)$ the set of points $(n_0, k_0) \in D'$ which generate trajectories converging to Λ, then $B(\Lambda) = D'$.

Since our attention will be mainly focused on the global properties of system (T, D'), in particular the boundaries of the strange attractor when it exists, we present some properties of non-invertible maps of the plane, and we describe a procedure to obtain the boundary of an invariant absorbing area. In fact when T is non-invertible, its global properties can be described by the method of critical curves, which may be used to obtain the minimal invariant absorbing area inside which asymptotic dynamics is confined.

System T is non-invertible since distinct points may have the same image, geometrically it folds and pleats the plane, so that two, or more, distinct points are mapped into the same point. As a consequence a point p has several distinct rank-1 preimages.

More formally our two-dimensional map $(n', k') = T(n, k)$ is non-invertible since the rank-1 preimages $(n, k) = T^{-1}(n', k')$ is more than one. In this case the plane can be subdivided into regions Z_j, $j \geqslant 0$, whose points have j distinct rank-1 preimages. Generally, as the point (n', k') varies, pairs of preimages appear or disappear as it crosses the boundaries separating different regions, hence such boundaries are characterized by the presence of at least two coincident (merging) preimages.

Following the notations of Mira et al. [28] and Abraham et al. [1], the critical curve of rank-1, denoted by LC, is defined as the locus of points having two, or more, coincident rank-1 preimages, located on a set denoted by LC_{-1} called curve of merging preimages. LC is the two-dimensional generalization of the notion of critical value of a one-dimensional map. Arcs of LC separate the plane into regions characterized by a different number of real preimages.

For the two-dimensional map (T, D'), that is an endomorphism, with f and g continuously differentiable, the locus LC_{-1} is generally given by the set of points such that $|J(n, k)| = 0$, where $J(n, k)$ is the Jacobian matrix of the map T.

In fact in any neighborhood of a point of LC_{-1} there are at least two distinct points which are mapped by T in the same point, hence the map is not locally invertible in points of LC_{-1}. More precisely, since T is locally an orientation preserving map near points (n, k) such that $|J(n, k)| > 0$ and an orientation reversing if $|J(n, k)| < 0$, then being T continuous the fold LC_{-1} is included in the set where $|J(n, k)|$ changes sign.

While considering system (T, D'), the Jacobian matrix is given by

$$J(n,k) = \begin{pmatrix} f'(n) & 0 \\ \frac{\partial g}{\partial n}(n,k) & \frac{\partial g}{\partial k}(n,k) \end{pmatrix} \tag{6}$$

so that

$$|J(n,k)| = \mu(1-2n)\left(\frac{1}{1+n}\left\{1-\delta+k^{\rho-1}(k^{\rho}+1)^{\frac{1-2\rho}{\rho}}[s_w+s_r k^{\rho}+\rho(s_r-s_w)]\right\}\right) \tag{7}$$

Observe that the Jacobian matrix is lower triangular so that it cannot have complex eigenvalues and thus the occurrence of regular oscillations, similar to those usually observed in Kaldor type models, is ruled out.

For a triangular map the following property holds.

Property 5.2. *The locus* LC_{-1} *of the phase plane is made up of curves* $LC_{-1}^{a_k}$ *and* LC_{-1}^{b} *such that*:

- $LC_{-1}^{a_k}$ *are vertical lines of equation* $n = c_{-1}^{a_k}$, *where* $c_{-1}^{a_k}$ *satisfy* $f'(c_{-1}^{a_k}) = 0$;
- LC_{-1}^{b} *is the locus* $\frac{\partial g}{\partial k}(n,k) = 0$.

Referring to our triangular map, the following proposition holds.

Proposition 5.3. *The locus* LC_{-1} *of the phase plane is made up of two curves* LC_{-1}^{a} *and* LC_{-1}^{b} *such that*:

(*i*) LC_{-1}^{a} *is the vertical line of equation* $n = \frac{1}{2}$
(*ii*) LC_{-1}^{b} *is the set of points* (n, k) *belonging to the horizontal lines* $k = k_m$ *and* $k = k_M$, *where* k_m *and* k_M *are the minimum and the maximum points of* g, *or* $LC_{-1}^{b} = \{\emptyset\}$.

In fact for the logistic map the critical value is $n' = c = \frac{\mu}{4}$ and the critical point is $n = c_{-1} = \frac{1}{2}$. A point $n' < c$ has two preimages and no preimages if $n' > c$ thus the region Z_2 is the set of points below c, the region Z_0 the set of points above c and the critical value c is the boundary separating the two regions. Such a point has two merging preimages $n_1 = n_2 = c_{-1} = \frac{1}{2}$.

About LC_{-1}^{b}, consider that the locus

$$\frac{\partial g}{\partial k}(n,k) = 0 \tag{8}$$

is given by the set of points $(n, \bar{k})$ such that $n \in R_+$ and $\bar{k}$ solves (8). Recall that function g can be bimodal or strictly monotonic in k. In the first case LC_{-1}^{b} is composed by two horizontal lines of equations $k = k_m$ and $k = k_M$ respectively. In the second case $LC_{-1}^{b} = \{\emptyset\}$.

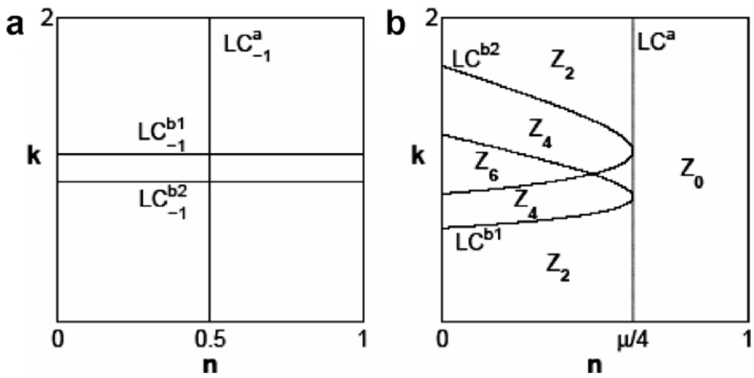

Fig. 2. (a) Critical curves of rank-0, LC_{-1} for system T and the following parameters values: $\rho = -30$, $\delta = 0.01$, $s_w = 0.1$, $s_r = 0.9$, $\mu = 2.5$. (b) Critical curves of rank-1, $LC = T(LC_{-1})$ for the same parameter values as in panel a. These curves separate the plane into the regions Z_6, Z_4, Z_2 and Z_0, whose points have different number of preimages.

In the following we assume $s_r > s_w$ and $\rho < 0$ small enough, so that map g is bimodal in k. The set of points for which the determinant of the Jacobian matrix vanishes is given by the union of three straight lines denoted by LC^a_{-1}, LC^{b1}_{-1} and LC^{b2}_{-1} in Fig. 2, panel a.

Being LC the rank-1 image of LC_{-1}, i.e. $LC = T(LC_{-1})$, we can give the following proposition.

Proposition 5.4. LC^a *is the vertical line of equation* $n = \frac{\mu}{4}$, *while* LC^b *is described by two parabola. The rank*-1 *image* LC *is the union of three branches*, $LC^a = T(LC^a_{-1}), LC^{b1} = T(LC^{b1}_{-1})$ *and* $LC^{b2} = T(LC^{b2}_{-1})$.

The three branches of the rank-1 image LC are shown in Fig. 2, panel b. The vertical line LC^a separates the region Z_0, whose points have no preimages, from the region with one or more preimages. In fact a point in Z_0 is such that $n > \frac{\mu}{4}$ so that no preimages are owned while considering f and consequently system T. Nevertheless, if $n < \frac{\mu}{4}$, so that we consider the region at the left of LC^a, these points admit at least two distinct preimages due to f.

Moreover we are considering parameter values such that $g(n, k)$ is bimodal in k, so that the one-dimensional map $k' = g(\bar{n}, k)$ is of the kind Z_1–Z_3–Z_1, where the three different sets are separated by $g(\bar{n}, k_m) = LC^{b1}$ and $g(\bar{n}, k_M) = LC^{b2}$,[9] see Fig. 3 representing function $g(\bar{n}, k)$ qualitatively for an assigned $n = \bar{n}$. In fact points k' with $k' < LC^{b1}$ or $k' > LC^{b2}$ have a unique preimage, points satisfying $LC^{b1} < k' < LC^{b2}$ have three distinct preimages, each of the points $k' = LC^{b1}$ and $k' = LC^{b2}$ has two preimages which merge in a critical point together with a second distinct preimage called *extra-preimage*.

As a consequence, if we consider the action of both components, system T is non-invertible and of (Z_6–Z_4–Z_2–Z_0)-type. In other words our system is such that the plane is divided into several unbounded open regions: a region Z_0 whose points have no preimages, a region Z_6 whose points generate six real rank-1 preimages, two regions Z_4 of points having four preimages and the remained regions are of Z_2-type. The boundaries of the regions Z_j, $j = 0, 2, 4, 6$ are branches of the rank-one critical curves $LC = LC^a \cup LC^{b1} \cup LC^{b2}$, see Fig. 2, panel b.

The critical sets of rank-h are the images of rank-h of LC_{-1} denoted by $LC_{h-1} = T^h(LC_{-1}) = T^{h-1}(LC)$ where $LC_0 = LC$. In order to define trapping regions of the phase plane, segments of critical curves of rank-h, $h = 0, 1, \ldots$, can be used. In fact an absorbing area A is a bounded region of the plane whose boundary is given by critical curve segments (i.e. segments of the critical curve LC and its images) such that a neighborhood $U \supset A$ exists whose points enter in A after a finite number of iterations and then never leave it, i. e. $T(A) \subseteq A$.

Following Mira et al. [28] a practical procedure to obtain the boundary of an absorbing area is now described. Starting from a portion of LC_{-1} (taken into the interesting region), its images of increasing rank are computed until a closed region is obtained. When such a region is mapped into itself, then it is an absorbing area A. The length of the initial segment has to be taken, in general, by a trial and error method. Once an absorbing area is found, in order to see if it is invariant, the same procedure must be repeated by taking only the portion $\gamma = A \cap LC_{-1}$ as the starting segment. If a natural m exists such that the union of the iterates of γ covers the whole boundary of A, then A is invariant.

In the following, some numerical simulations are presented and the absorbing area is depicted.

[9] Notice that in this case LC^{b1} and LC^{b2} reduce both to a singleton.

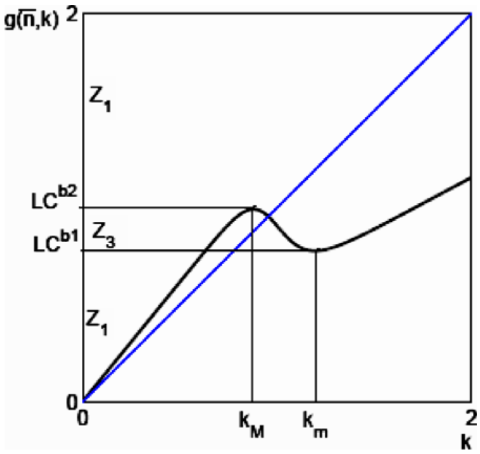

Fig. 3. Scheme of the graph of map $g(\bar{n}, k)$ when it is bimodal ($\rho < 0$ is low enough).

5.1. Numerical simulations

In this section we describe the bifurcations which increase the complexity of the asymptotic dynamic behaviour of the system. Consider set D' and assume Property 5.2 not being fulfilled, that is $s_r > s_w$ and $\rho < \bar{\rho}$. Thus we are referring to the case in which shareholders save more than workers and the elasticity of substitution between production factors is low enough. In this case g is bimodal in k and system (T, D') admits a unique fixed point $E^* = (n^*, k^*)$. A feasible trajectory may converge to the positive steady state E^* or to other more complex attractor inside D'.

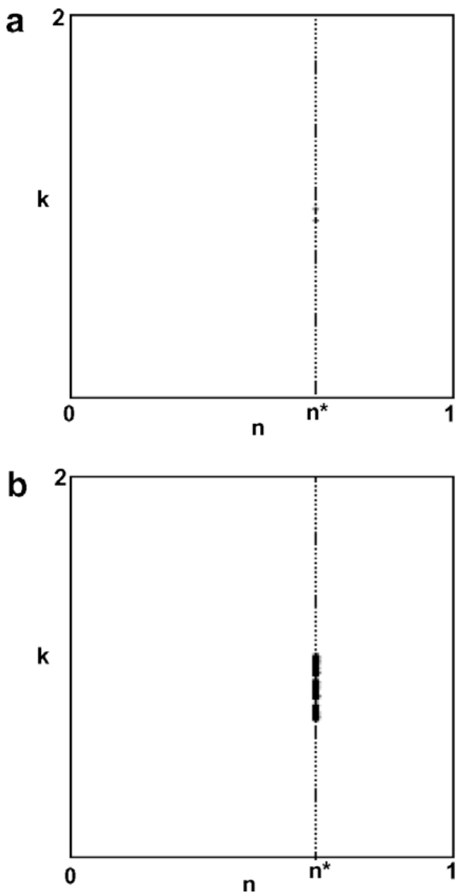

Fig. 4. (a) Two period cycle for $\rho = -20$, $\delta = 0.01$, $s_w = 0.1$, $s_r = 0.9$, $\mu = 2.8$. (b) Strange attractor for $\rho = -50$ and the other parameter values as in panel a.

We perform some numerical simulations in order to better understand the final dynamics of our model. We fix the values of the following parameters: $\delta = 0.01$, $s_w = 0.1$, $s_r = 0.9$. Consequently, observe that only the two parameters ρ and μ must be analyzed. In fact, recall that both such parameters play an important role in our model for complicated dynamics to be observed. We consider initial conditions belonging to D' and observe the attractor owned by T.

We first fix μ at the value 2.8. Obviously the logistic map converges to the fixed point n^* so that system (T, D') has a global attractor situated on the line $n = n^*$. The structure of the attractor depends on the dynamical property of the one-dimensional limiting map $g_{n^\star}$ and hence on the value of ρ. Observe Fig. 4. For different values of ρ, the attractor is a two period cycle or a strange attractor (the whole segment $\{(n^*, k):k \in [\mathrm{LC}^{b1}, \mathrm{LC}^{b2}]\}$).

As it can be noticed from the previous figure, the attractor Λ existing inside D' changes its structure for decreasing values of the parameter ρ. For $\rho = -20$ the attractor is a period-two cycle to which all the trajectories starting inside D' converge. As ρ decreases the cycle loses stability via flip bifurcation of the map $g_{n^\star}$ until a chaotic attractor is created. Anyway since the first component of system T converges to the fixed point, the attractor belongs to the line $n = n^*$.

In Fig. 5, we show different kinds of attractors that can be obtained when varying the parameter μ as $\rho = -50$.

Remember Fig. 1a, the attractor Λ is a cycle of period 4. As μ is increased, a sequence of period-doubling bifurcations gives rise to cycles of period 2^k and then to cyclic chaotic attractors as the 4-cyclic chaotic attractor in Fig. 5, panel a. Such a picture is quite interesting since we assume $\mu = 3.6$ so that the attractor of the logistic map belongs to two disjoint intervals and consequently the attractor of T is now given by the union of four trapping sets.

In this situation the long-run behaviour of the system is characterized by cyclical behaviour of order 4, but in each period the exact state cannot be predicted. We observe a strange attractor made up of four pieces which increase in size as μ increases. If μ is further increased the four cyclic chaotic attractor gives rise to a connected chaotic attractor as in panel b after a contact of the four chaotic areas: a cyclic (although chaotic) behaviour is replaced by a totally erratic evolution that covers a wide area of the phase space of our dynamical system.

Finally in Fig. 6 we assume $\rho = -10$ and observe again strange attractors for two different values of μ.

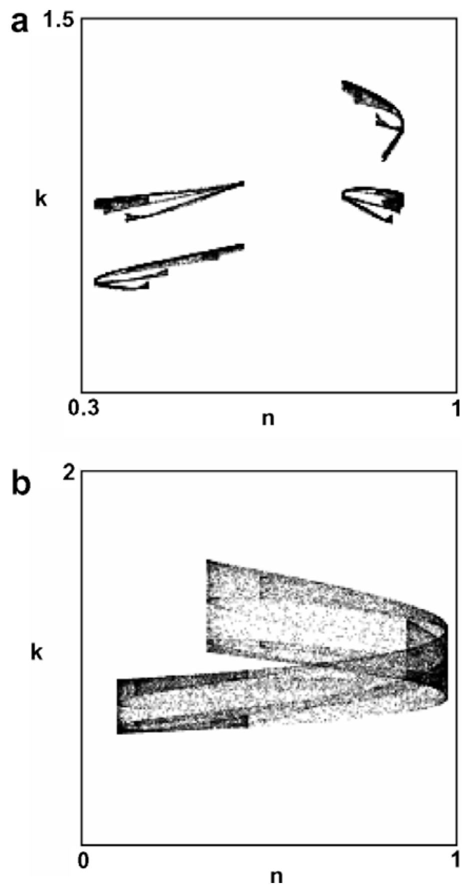

Fig. 5. (a) Four cyclic chaotic attractor for $\rho = -50$, $\delta = 0.01$, $s_w = 0.1$, $s_r = 0.9$, $\mu = 3.6$. (b) Chaotic attractor for $\mu = 3.9$ and the other parameters as in panel a.

Our simulations show that for $\mu \in (1, 4)$ the attractor Λ can be quite complicated. When μ is increased, so that the bifurcation value $\mu = 4$ is crossed, a global bifurcation occurs and the attractor Λ disappears. This bifurcation is characterized by a contact between the boundary of D', namely $\partial D'$, and the critical curve $n = \frac{\mu}{4}$.

In order to obtain the boundary of the chaotic area, namely $\partial\, \Gamma$, we use the procedure previously defined that is, being $\gamma = \Gamma \cap \mathrm{LC}_{-1}$ the portion of critical curve of rank-0 inside Γ, then for a suitable integer r

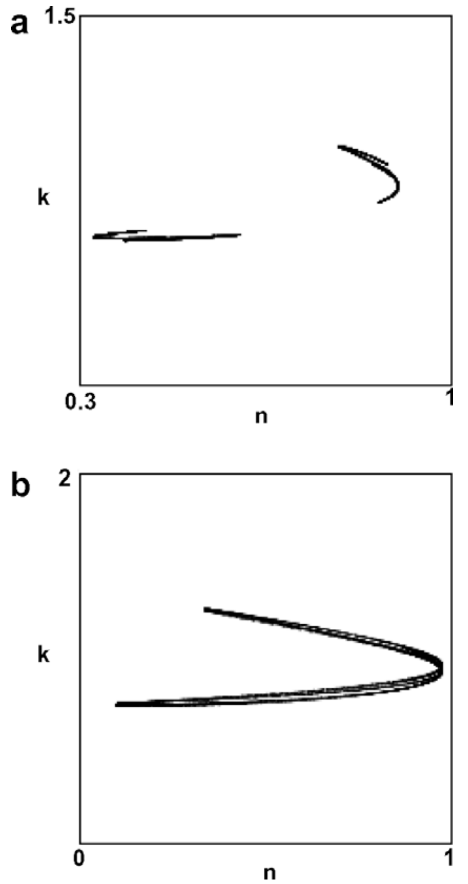

Fig. 6. (a) Two pieces chaotic attractor for $\rho = -10$, $\delta = 0.01$, $s_w = 0.1$, $s_r = 0.9$, $\mu = 3.6$. (b) Connected chaotic attractor for $\mu = 3.9$ and the other parameters as in panel a.

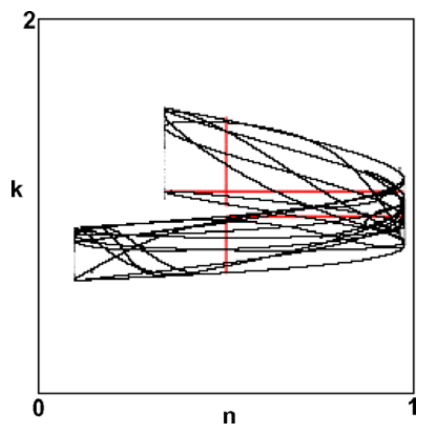

Fig. 7. Boundary of the attractor in Fig. 5, panel b, Set γ is depicted in red. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$\partial\Gamma \subseteq \cup_{h=1}^{r} T^{h}(\gamma)$.

An example is shown in Fig. 7 where we numerically determine the boundaries of the attractors shown in Fig. 5, panel b.

The boundary of the chaotic area is obtained by the images, up to rank 6, of the portion γ of LC_{-1} depicted in red. It is worth noticing that the critical curves of increasing rank not only give the boundary of the strange attractor, but also the regions of greater density of points, i.e. the regions that are more frequently visited by points of the generic trajectory in the invariant area Γ.

6. Conclusions

In this paper we investigated the global properties of the Solow growth model with differential saving and logistic population growth rate. We first proved that our model admits a compact global attractor inside which asymptotic states are confined. We then described its structure as parameters of the system vary.

We observed more complex features as μ is increased (so that the amplitude of fluctuations in population growth rate increases) or ρ is decreased so that the capital income monotonicity property does not hold. Leontief production function is the extreme example of such consideration.

The critical curves method have been used to obtain the boundary of compact trapping regions, called absorbing areas, inside which cyclic and erratic fluctuations are performed. The new ingredient of our model, i.e. the logistic population growth rate, proved convincing in order to consider possible fluctuations in the population growth rate. We showed how complex dynamics depends on such fluctuations as well as the fall in the capital income. A further interesting question is what happens introducing population dynamics with an economic feedback, i.e. where f is a function of both n and k, but we leave such approach to future research lines.

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