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CONTINUOUS DEPENDENCE OF ATTRACTORS ON PARAMETERS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS AND INFINITE ITERATED FUNCTION SYSTEMS

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Abstract. The paper is dedicated to the study of the problem of continuous dependence of compact global attractors on parameters of non-autonomous dynamical systems and infinite iterated function systems (IIFS). We prove that if a family of non-autonomous dynamical systems $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ depending on parameter $\lambda \in \Lambda$ is uniformly contracting (in the generalized sense), then each system of this family admits a compact global attractor J^λ and the mapping $\lambda \to J^\lambda$ is continuous with respect to the Hausdorff metric. As an application we give a generalization of well known Theorem of Bransley concerning the continuous dependence of fractals on parameters.

1. **Introduction.** The aim of this paper is the study of the problem of existence of compact global attractors of non-autonomous dynamical systems and their continuous dependence on parameters. The problem of the upper semi-continuous dependence on parameters of global attractors of dynamical systems is well studied (both autonomous and non-autonomous, see for example Caraballo, Langa and Robinson [3], Caraballo and Langa [4], Cheban [6, 7] Hale and Raugel [15],Hale [16] and also see the bibliography therein). The problem of the lower semi-coninuous dependence on parameters of global attractors is less extensively studied. Note, for example, the works of Dupaix, Hilhorst and Kostin [11], Elliott and Kostin [13], Hale [16], Hale and Raugel [17], Kapitanskii and Kostin [20], Kostin [21], Li and Kloeden [22], Stuart and Humphries [28] and the bibliography therein.

The paper is dedicated to the study of the problem of continuous dependence of compact global attractors on parameters of non-autonomous dynamical systems and infinite iterated function systems (IIFS). We prove that if a family of non-autonomous dynamical systems $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ depending on parameter $\lambda \in \Lambda$ is uniformly contracting (in the generalized sense), then each system of this family admits a compact global attractor J_λ and the mapping $\lambda \to J_\lambda$ is continuous with respect to the Hausdorff metric. As an application we give a generalization of

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well known Theorem of Bransley concerning the continuous dependence of fractals on parameters.

This paper is organized as follows.

In Section 2 we give some notions and facts from the theory of set-valued dynamical systems which we use in our paper.

Section 3 is dedicated to the study of upper semi-continuous (generally speaking set-valued) invariant sections of non-autonomous dynamical systems. They play a very important role in the study of non-autonomous dynamical systems. We give the sufficient conditions which guarantee the existence of a unique globally exponentially stable invariant section. The main result of this paper is Theorem 3.

We give in section 4 a new approach to the study of discrete inclusions (DI) which is based on non-autonomous dynamical systems (See also our previous works [8, 9, 10], where we study the IFSs (both linear [8, 9] and nonlinear [10] cases) in the framework of non-autonomous dynamical systems (cocycles).). We show that every DI in a natural way generates some non-autonomous dynamical system (cocycle), which plays an important role in its study (see Sections 7 and 8).

In section 5 we study some properties of Lipschitz maps. We introduce the notion of spectral radius for Lipschitzian maps and we give the necessary and sufficient conditions that a Lipschitzian mapping is contracting in the generalized sense in the term of its spectral radius (Lemma 3).

In Section 6 we study the relation between a compact global attractor of cocycle and the skew-product dynamical system (respectively, set-valued dynamical system) associated by the given cocycle.

Section 7 is dedicated to the study of the problem of continuous dependence of attractors of infinite iterated function systems. We give a generalization of well known Theorem of Bransley concerning the continuous dependence of fractals on parameters (Theorem 10).

2. **Set-Valued dynamical systems and their compact global attractors.**

Let (X,ρ) be a complete metric space, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a subsemi-group of $\mathbb{S}$. If $A \subseteq X$ and $x \in X$, then we denote by $\rho(x,A)$ the distance from the point x to the set A, i.e. $\rho(x,A) = \inf\{\rho(x,a) : a \in A\}$. We denote by $B(A,\varepsilon)$ an ε-neighborhood of the set A, i.e. $B(A,\varepsilon) = \{x \in X : \rho(x,A) < \varepsilon\}$, by $K(X)$ we denote the family of all non-empty compact subsets of X. For every point $x \in X$ and number $t \in \mathbb{T}$ we put in correspondence a closed compact subset $\pi(t,x) \in K(X)$. So, if $\pi(P,A) = \bigcup\{\pi(t,x) : t \in P, x \in A\}(P \subseteq \mathbb{T})$, then

(i) $\pi(0,x) = x$ for all $x \in X$;

(ii) $\pi(t_2, \pi(t_1,x)) = \pi(t_1+t_2, x)$ for all $x \in X$ and $t_1, t_2 \in \mathbb{T}$;

(iii) $\lim\limits_{x\to x_0, t\to t_0} \beta(\pi(t,x), \pi(t_0,x_0)) = 0$ for all $x_0 \in X$ and $t_0 \in \mathbb{T}$, where $\beta(A,B) = \sup\{\rho(a,B) : a \in A\}$ is a semi-deviation of the set $A \subseteq X$ from the set $B \subseteq X$.

In this case it is said (see, for example, [27] and [23] and the bibliography therein) that there is defined a set-valued semi-group dynamical system.

Let $\mathbb{T} = \mathbb{S}$ and be fulfilled the next condition:

(i) if $p \in \pi(t,x)$, then $x \in \pi(-t,p)$ for all $x, p \in X$ and $t \in \mathbb{T}$.

Then it is said that there is defined a set-valued group dynamical system $(X,\mathbb{T},\pi)$ or a bilateral (two-sided) dynamical system.

Definition 1. Let $\mathbb{T}' \subset \mathbb{S}$ ($\mathbb{T} \subset \mathbb{T}'$). A continuous mapping $\gamma_x : \mathbb{T} \to X$ is called a motion of the set-valued dynamical system $(X, \mathbb{T}, \pi)$ issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on $\mathbb{T}'$, if

a. $\gamma_x(0) = x$;
b. $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ for all $t_1, t_2 \in \mathbb{T}'$ $(t_2 > t_1)$.

The set of all motions of $(X, \mathbb{T}, \pi)$, passing through the point x at the initial moment $t = 0$ is denoted by $\mathcal{F}_x(\pi)$ and we define $\mathcal{F}(\pi) := \bigcup\{\mathcal{F}_x(\pi) \mid x \in X\}$ (or simply $\mathcal{F}$).

Definition 2. Any trajectory $\gamma \in \mathcal{F}(\pi)$ defined on $\mathbb{S}$ is called a full (entire) trajectory of the dynamical system $(X, \mathbb{T}, \pi)$.

Denote by $\Phi(\pi)$ the set of all full trajectories of the dynamical system $(X, \mathbb{T}, \pi)$ and $\Phi_x(\pi) := \mathcal{F}_x(\pi) \bigcap \Phi(\pi)$.

Theorem 1. [27] *Let $(X, \mathbb{T}, \pi)$ be a semi-group dynamical system and X be a compact and invariant set (i.e. $\pi^t X = X$ for all $t \in \mathbb{T}$, where $\pi^t := \pi(t, \cdot)$). Then*

(i) *$\mathcal{F}(\pi) = \Phi(\pi)$, i.e. every motion $\gamma \in \mathcal{F}_x(\pi)$ can be extended on $\mathbb{S}$ (this means that there exists $\tilde{\gamma} \in \Phi_x(\pi)$ such that $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in \mathbb{T}$);*
(ii) *there exists a group (generally speaking set-valued) dynamical system $(X, \mathbb{S}, \tilde{\pi})$ such that $\tilde{\pi}|_{\mathbb{T}\times X} = \pi$.*

Definition 3. A system $(X, \mathbb{T}, \pi)$ is called [5, 7] compactly dissipative, if there exists a nonempty compact $K \subseteq X$ such that

$$\lim_{t\to+\infty} \beta(\pi^t M, K) = 0;$$

for all $M \in K(X)$, where $\pi^t M := \pi(t, M)$.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and K be a compact set attracting every compact subset of X. Let us set

$$J := \omega(K) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \pi^\tau K}. \tag{1}$$

It can be shown [5, 7] that the set J defined by equality (1) does not depends on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J is called a center of Levinson of the compact dissipative system $(X, \mathbb{T}, \pi)$.

Theorem 2. [5, 7] *If $(X, \mathbb{T}, \pi)$ is a compactly dissipative dynamical system and J is its center of Levinson, then :*

(i) *J is invariant, i.e. $\pi^t J = J$ for all $t \in \mathbb{T}$;*
(ii) *J is orbitally stable, i.e. for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\beta(\pi(t, x), J) < \varepsilon$ for all $t \geq 0$;*
(iii) *J is an attractor of the family of all compact subsets of X;*
(iv) *J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$.*

3. **Upper semi-continuous invariant sections of non-autonomous dynamical systems and their continuous dependence on parameters.** In this section we study the upper semi-continuous (generally speaking set-valued) invariant sections of non-autonomous dynamical systems. They play a very important role in the study of non-autonomous dynamical systems. We give the sufficient conditions

which guarantee the existence of a unique globally exponentially stable invariant section and their continuous dependence on parameters.

Lemma 1. *Let X and Λ be complete metric spaces. Let $(X,\mathbb{T},\pi_\lambda)$ ($\lambda\in\Lambda$) be a family of dynamical systems satisfying the following conditions:*

(i) *the family of dynamical systems $(X,\mathbb{T},\pi_\lambda)$ ($\lambda\in\Lambda$) is uniformly contracting, i.e. there exist two positive numbers $\mathcal{N}$ and ν such that $\rho(\pi_\lambda(t,x_1),\ \pi_\lambda(t, x_2))\leq \mathcal{N}e^{-\nu t}\rho(x_1,x_2)$ for all $\lambda\in\Lambda, t\in\mathbb{T}$ and $x_1,x_2\in X$;*

(ii) *for each $t\in\mathbb{T}$ the mapping $(\lambda,x)\mapsto\pi_\lambda(t,x)$ is continuous.*

Then for each $\lambda\in\Lambda$ the dynamical system $(X,\mathbb{T},\pi_\lambda)$ admits a unique stationary point p_λ and the mapping $\lambda\mapsto p_\lambda$ is continuous.

Proof. Let $\Lambda^{'}$ be a compact subset of Λ. Denote by $C(\Lambda^{'},X)$ the space of all continuous functions $\varphi:\Lambda^{'}\mapsto X$ with distance $r(\varphi_1,\varphi_2):=\max\{\rho(\varphi_1(\lambda),\varphi_2(\lambda)) : \lambda\in\Lambda^{'}\}$. $(C(\Lambda^{'},X),r)$ is a complete metric space. Note that under the conditions of the lemma if $\varphi\in C(\Lambda^{'},X)$ then also $\psi_t\in C(\Lambda^{'},X)$, where $\psi_t(\lambda):=\pi_\lambda(t,\varphi(\lambda))$ for all $\lambda\in\Lambda^{'}$, where $t\in\mathbb{T}$. Denote by $S^t_{\Lambda'}$ the mapping from $C(\Lambda^{'},X)$ into itself defined by equality $(S^t_{\Lambda'}\varphi)(\lambda):=\pi_\lambda(t,\varphi(\lambda))$ for all $t\in\mathbb{T}$ and $\lambda\in\Lambda^{'}$. It is easy to check that $\{S^t_{\Lambda'}\}_{t\in\mathbb{T}}$ is a commutative semi-group (with respect to composition) and $r(S^t_{\Lambda'}\varphi_1,S^t_{\Lambda'}\varphi_2)\leq\mathcal{N}e^{-\nu t}r(\varphi_1,\varphi_2)$ for all $t\in\mathbb{T}$ and $\varphi_1,\varphi_2\in C(\Lambda^{'},X)$. Hence there exists a unique common fix point $\varphi_{\Lambda'}\in C(\Lambda^{'},X)$ of semi-group $\{S^t_{\Lambda'}\}_{t\in\mathbb{T}}$. In particularly $\pi_\lambda(t,\varphi_{\Lambda'}(\lambda))=\varphi_{\Lambda'}(\lambda)$ for all $\lambda\in\Lambda^{'}$, i.e. $p_\lambda:=\varphi_{\Lambda'}(\lambda)$ is a unique stationary point of dynamical system $(X,\mathbb{T},\pi_\lambda)$ and the mapping $\lambda\mapsto p_\lambda$ from $\Lambda^{'}$ into X is continuous.

Thus we have a family of commutative semi-groups $\{S^t_{\Lambda'}\}_{t\in\mathbb{T}}$ depending on parameter $\Lambda^{'}\in K(\Lambda)$. It is easy to check that the following statements are true:

a. for each $\Lambda^{'}\in K(\Lambda)$ the commutative semi-group $\{S^t_{\Lambda'}\}_{t\in\mathbb{T}}$ admits a unique stationary point $\varphi_{\Lambda'}\in C(\Lambda^{'},X)$;

b. if $\Lambda^{'}\subseteq\Lambda''$ then $\tilde{\varphi}_{\Lambda''}=\varphi_{\Lambda'}$, where $\tilde{\varphi}_{\Lambda''}$ is the restriction on $\Lambda^{'}$ of function $\varphi_{\Lambda''}$;

c. $\varphi_{\Lambda'}(\lambda)=\varphi_{\Lambda''}(\lambda)$ for all $\lambda\in\Lambda^{'}\cap\Lambda^{''}$ and $\Lambda^{'},\Lambda^{''}\in K(\Lambda)$.

Denote by $C(\Lambda,X)$ the space of all continuous functions $\varphi:\Lambda\mapsto X$ equipped with compact-open topology (the topology of convergence uniform on every compact subset $\Lambda^{'}\subseteq\Lambda$). Let S^t be the mapping from $C(\Lambda,X)$ into itself defined by equality $(S^t\varphi)(\lambda):=\pi_\lambda(t,\varphi(\lambda))$ for all $t\in\mathbb{T}$ and $\lambda\in\Lambda$. It is easy to check that $\{S^t\}_{t\in\mathbb{T}}$ is a commutative semi-group (with respect to composition). We define now the mapping $\varphi:\Lambda\mapsto X$ as follow:

$$\varphi(\lambda):=\varphi_{\Lambda'}(\lambda), \tag{2}$$

where $\Lambda^{'}\in K(\Lambda)$ is an arbitrary compact subset of Λ containing λ. According to properties a.-c. by equality (2) a function $\varphi\in C(\Lambda,X)$ is correctly defined and it is a unique stationary point of the semi-group $\{S^t\}_{t\in\mathbb{T}}$. This means that $S^t\varphi=\varphi$ for all $t\in\mathbb{T}$ or equivalently $\pi_\lambda(t,\varphi(\lambda))=\varphi(\lambda)$ for all $\lambda\in\Lambda$ and $t\in\mathbb{T}$, i.e. the point $p_\lambda:=\varphi(\lambda)$ is a unique stationary point of dynamical system $(X,\mathbb{T},\pi_\lambda)$ and the mapping $\lambda\mapsto p_\lambda$ is continuous. □

Remark 1. Lemma 1 is also true if

(i) we replace the condition of uniform contraction by the following weaker condition: for each compact subset $\Lambda^{'} \subseteq \Lambda$ there are two positive numbers $\mathcal{N}_{\Lambda'}$ and $\nu_{\Lambda'}$ such that $\rho(\pi_\lambda(t,x_1),\pi_\lambda(t,x_2)) \le \mathcal{N}_{\Lambda'} e^{-\nu_{\Lambda'} t}\rho(x_1,x_2)$ for all $\lambda \in \Lambda^{'}, t \in \mathbb{T}$ and $x_1, x_2 \in X$;

(ii) we consider in place of family of dynamical systems $(X,\mathbb{T},\pi_\lambda)_{\lambda\in\Lambda}$ an arbitrary family of commutative semi-groups $\{\pi^t_\lambda\}_{t\in\mathbb{T}}$ $(\lambda \in \Lambda)$ with conditions:
   (a) for each $t \in \mathbb{T}$ the mapping $(\lambda, x) \mapsto \pi^t_\lambda x$ is continuous;
   (b) there are two positive numbers $\mathcal{N}$ and ν such that $\rho(\pi_\lambda(t,\ x_1),\ \pi_\lambda(t,\ x_2)) \le \mathcal{N} e^{-\nu t}\rho(x_1,x_2)$ for all $\lambda \in \Lambda, t \in \mathbb{T}$ and $x_1, x_2 \in X$.

Definition 4. Let X be a metric space and Y be a topological space. The set-valued mapping $\gamma : Y \to K(X)$ is said to be upper semi-continuous (or β-continuous), if $\lim\limits_{y\to y_0} \beta(\gamma(y),\gamma(y_0)) = 0$ for all $y_0 \in Y$.

Definition 5. Let (X,h,Y) be a fiber space, i.e. $h : X \mapsto Y$ is a continuous mapping from X onto Y. The mapping $\gamma : Y \to K(X)$ is called a section (selector) of the fiber space (X,h,Y), if $h(\gamma(y)) = y$ for all $y \in Y$.

Remark 2. Let $X := W \times Y$. Then $\gamma : Y \to X$ is a section of the fiber space (X,h,Y) $(h := pr_2 : X \to Y)$, if and only if $\gamma = (\psi, Id_Y)$ where $\psi : W \to K(W)$.

Definition 6. Let $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ be two dynamical systems. The mapping $h : X \to Y$ is called a homomorphism (respectively isomorphism) of the dynamical system $(X,\mathbb{T}_1,\pi)$ on $(Y,\mathbb{T}_2,\sigma)$, if the mapping h is continuous (respectively homeomorphic) and $h(\pi(x,t)) = \sigma(h(x),t)$ $(\ t \in \mathbb{T}_1,\ x \in X)$.

Remark 3. In this work we show that every IFS generates some non-autonomous dynamical system (see Section 4 and also [10]). Many examples of non-autonomous dynamical systems, generated by non-autonomous differential/difference equations (ODEs, PDEs and functional-differential equations) can be found by the reader, for example, in the books [7] and [24].

Definition 7. A triplet $\langle (X,\mathbb{T}_1,\pi),\ (Y,\mathbb{T}_2,\sigma),\ h\rangle$, where h is a homomorphism of $(X,\mathbb{T}_1,\pi)$ on $(Y,\mathbb{T}_2,\sigma)$ and (X,h,Y) is a fiber space, is called a non-autonomous dynamical system.

Definition 8. A mapping $\gamma : Y \to X$ is called an invariant section of the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$, if it is a section of the fiber space (X,h,Y) and $\gamma(Y)$ is an invariant subset of the dynamical system $(X,\mathbb{T},\pi)$ (or, equivalently, $\pi^t\gamma(y) = \gamma(\sigma^t y)$ for all $t \in \mathbb{T}$ and $y \in Y$).

Denote by $\alpha : K(X) \times K(X) \to \mathbb{R}_+$ the Hausdorff distance on $K(X)$, i.e.

$$\alpha(A,B) := \max(\beta(A,B),\beta(B,A)).$$

Theorem 3. *Let Λ be a metric space, $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$ $(\lambda \in \Lambda)$ be a family of non-autonomous dynamical system and suppose the following conditions are fulfilled:*

(i) *the space Y is compact;*

(ii) *Y is invariant, i.e. $\sigma^t Y = Y$ for all $t \in \mathbb{T}_2$;*

(iii) *the non-autonomous dynamical systems $\langle (X,\mathbb{T}_1,\pi_\lambda),(Y,\mathbb{T}_2,\sigma),h\rangle$ are equicontracting in the extended sense, i.e. there exist positive numbers N and ν such that*

$$\rho(\pi_\lambda(t,x_1),\pi_\lambda(t,x_2)) \le N e^{-\nu t}\rho(x_1,x_2) \tag{3}$$

for all $\lambda \in \Lambda$, $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$ and $t \in \mathbb{T}_1$;

(iv) *for each* $t \in \mathbb{T}_1$ *the mapping* $(\lambda, x) \to \pi_\lambda(t,x)$ *from* $\Lambda \times X$ *into* X *is continuous;*
(v) $\Gamma(Y,X) = \{\gamma \mid \gamma : Y \to K(X)$ *is a set-valued* β*–continuous mapping and* $h(\gamma(y)) = y$ *for all* $y \in Y\} \neq \emptyset$.

Then

(i) *for each* $\lambda \in \Lambda$ *there exists a unique invariant section* $\gamma_\lambda \in \Gamma(Y,X)$ *of the non-autonomous dynamical system* $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$;
(ii) *the non-autonomous dynamical system* $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ *is compactly dissipative (i.e.* $(X, \mathbb{T}_1, \pi_\lambda)$ *is compactly dissipative) and its Levinson's center* $J^\lambda = \gamma_\lambda(Y)$;
(iii) $\pi^t_\lambda J^\lambda_y = J^\lambda_{\sigma(t,y)}$ *for all* $t \in \mathbb{T}_1$ *and* $y \in Y$;
(iv) *the mapping* $\lambda \to \gamma_\lambda$ *is continuous, i.e.*

$$\lim_{\lambda \to \lambda_0} \sup_{y \in Y} \alpha(\gamma_\lambda(y), \gamma_{\lambda_0}(y)) = 0;$$

(v) *if* $(Y, \mathbb{T}_2, \sigma)$ *is a group-dynamical system (i.e.* $\mathbb{T}_2 = \mathbb{S}$*), then the unique invariant section* γ_λ *of the non-autonomous dynamical system* $\langle (X, \mathbb{T}_1, \pi_\lambda), (Y, \mathbb{T}_2, \sigma), h\rangle$ *is one-valued (i.e.* $\gamma_\lambda(y)$ *consists a single point for any* $y \in Y$*) and*

$$\rho(\pi_\lambda(t,x), \pi_\lambda(t, \gamma_\lambda(h(x)))) \leq N e^{-\nu t} \rho(x, \gamma_\lambda(h(x))) \tag{4}$$

for all $x \in X$ *and* $t \in \mathbb{T}$.

Proof. Since the space Y is compact and invariant, then according to Theorem 1 the semi-group dynamical system $(Y, \mathbb{T}, \sigma)$ can be prolonged to a group set-valued dynamical system $(Y, \mathbb{S}, \tilde{\sigma})$ (this means that $\tilde{\sigma}(s,y) = \sigma(s,y)$ for all $(s,y) \in \mathbb{T} \times Y$).

Let $\alpha : K(X) \times K(X) \to \mathbb{R}_+$ be the Hausdorff's distance on $K(X)$ and $d : \Gamma(Y,X) \times \Gamma(Y,X) \to \mathbb{R}_+$ be the function defined by the equality

$$d(\gamma_1, \gamma_2) := \sup_{y \in Y} \alpha(\gamma_1(y), \gamma_2(y)). \tag{5}$$

Note that (5) defines a complete distance on $\Gamma(Y,X)$ (see [10]).

For $t \in \mathbb{T}_1$ and $\lambda \in \Lambda$, by S^t_λ we denote the mapping of $\Gamma(Y,X)$ into itself defined by the equality $(S^t_\lambda \gamma)(y) = \pi_\lambda(t, \gamma((\sigma^t)^{-1} y))$ for all $t \in \mathbb{T}_1$, $y \in Y$ and $\gamma \in \Gamma(Y,X)$. It is easy to see that $S^t_\lambda \gamma \in \Gamma(Y,X)$, $S^t_\lambda S^\tau_\lambda = S^{t+\tau}_\lambda$ for all $t, \tau \in \mathbb{T}_1$ and $\gamma \in \Gamma(Y,X)$ and, hence, $\{S^t_\lambda\}_{t \in \mathbb{T}_1}$ forms a commutative semi-group. We will show that

$$d(S^t_\lambda \gamma_1, S^t_\lambda \gamma_2) \leq \mathcal{N} e^{-\nu t} d(\gamma_1, \gamma_2) \tag{6}$$

for all $t \in \mathbb{T}_1$ and $\gamma_i \in \Gamma(Y,X)$ $(i = 1,2)$. In fact. To prove the inequality (6) it is sufficient to show that

$$\alpha(\pi^t_\lambda \gamma_1(\sigma^{-t} y), \pi^t_\lambda \gamma_2(\sigma^{-t} y) \leq \mathcal{N} e^{-\nu t} d(\gamma_1, \gamma_2) \tag{7}$$

for all $y \in Y$, where $\sigma^{-t} y := \{q \in Y \mid \sigma(t,q) = y\}$.

Let $v \in \pi^t_\lambda \gamma_2(\sigma^{-t} y)$ be an arbitrary element, then there is $q \in \sigma^{-t} y$ and $x_2(y) \in \gamma_2(q)$ so that $v = \pi^t_\lambda x_2(y)$. We choose $x_1(y) \in \gamma_1(q)$ such that

$$\rho(x_1(y), x_2(y)) \leq \alpha(\gamma_1(q), \gamma_2(q)) \leq d(\gamma_1, \gamma_2) \tag{8}$$

(by compactness of $\gamma_i(q)$ $(i = 1,2)$ obviously such an $x_1(y)$ exists there and additionally $h(x_1(y)) = h(x_2(y)) = q$). Then we have

$$\rho(\pi^t_\lambda x_1(y), \pi^t_\lambda x_2(y)) \leq \mathcal{N} e^{-\nu t} \rho(x_1(y), x_2(y)) \leq \mathcal{N} e^{-\nu t} d(\gamma_1, \gamma_2),$$

i.e. for all $v \in \pi^t_\lambda \gamma_2(\sigma^{-t} y)$ there exists $u := \pi^t x_1(y) \in \pi^t_\lambda \gamma_1(\sigma^{-t} y)$ so that $\rho(u,v) \leq \mathcal{N} e^{-\nu t} d(\gamma_1, \gamma_2)$. This means that $\beta(\pi^t_\lambda \gamma_1(\sigma^{-t} y), \pi^t_\lambda \gamma_2(\sigma^{-t} y)) \leq \mathcal{N} e^{-\nu t} d(\gamma_1, \gamma_2)$. Analogously, the inequality $\beta(\pi^t_\lambda \gamma_2(\sigma^{-t} y), \pi^t_\lambda \gamma_1(\sigma^{-t} y)) \leq \mathcal{N} e^{-\nu t}\ d(\gamma_1, \gamma_2)$ can be established and, consequently, $\alpha(\pi^t_\lambda\gamma_1(\sigma^{-t}y),\pi^t_\lambda\gamma_2(\sigma^{-t}y))\le\mathcal{N}e^{-\nu t}d(\gamma_1,\gamma_2)$ for all $y\in Y$ and $t\in\mathbb{T}_1$. Thus the inequality (7) is established.

We will show now that for each $t_0\in\mathbb{T}_1$ the mapping $(\lambda,\gamma)\to S^{t_0}_\lambda\gamma$ from $\Lambda\times\Gamma(Y,X)$ into $\Gamma(Y,X)$ is continuous. In fact. Let $\lambda_k\to\lambda_0$ and $\gamma_k\to\gamma_0$. We shall prove that $S^{t_0}_{\lambda_k}\gamma_k\to S^{t_0}_{\lambda_0}\gamma_0$ in the space Γ. Denote by

$$m(\lambda):=\sup_{x\in\gamma_0(Y)}\rho(\pi^{t_0}_\lambda x,\pi^{t_0}_{\lambda_0}x) \tag{9}$$

and note that $m(\lambda)\to 0$ as $\lambda\to\lambda_0$. If we suppose that it is not true, then there are $\varepsilon_0>0,\lambda_k\to\lambda_0$ and $x_k\to x_0$ $(x_k\in\gamma_0(Y))$ such that

$$\rho(\pi^{t_0}_{\lambda_k}x_k,\pi^{t_0}_{\lambda_0}x_k)\ge\varepsilon_0. \tag{10}$$

Passing to the limit in (10) as $k\to+\infty$ we obtain $\varepsilon_0\le 0$. The obtained contradiction shows that $m(\lambda)\to 0$ as $\lambda\to\lambda_0$.

Let $y\in Y$ and $v\in\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y)$, then there are $q\in\sigma^{-t_0}y$ and $x\in\gamma_0(q)$ such that $v=\pi^{t_0}_{\lambda_0}x$. Denote by $u:=\pi^{t_0}_\lambda x$, then we have

$$\rho(u,v)=\rho(\pi^{t_0}_\lambda x,\pi^{t_0}_{\lambda_0}x)\le\sup_{x\in\gamma_0(Y)}\rho(\pi^{t_0}_\lambda x,\pi^{t_0}_{\lambda_0}x)=m(\lambda). \tag{11}$$

From the inequality (11) it follows $\beta(\pi^{t_0}_\lambda\gamma_0(\sigma^{-t_0}y),\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y))\le m(\lambda)$. Analogously one can establish the inequality $\beta(\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y),\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y))\le m(\lambda)$ and, consequently,

$$\alpha(\pi^{t_0}_\lambda\gamma_0(\sigma^{-t_0}y),\pi^{t_0}_{\lambda_0}\gamma_0(\sigma^{-t_0}y))\le m(\lambda) \tag{12}$$

for all $y\in Y$ and $\lambda\in\Lambda$. From (12) it follows that

$$d(S^{t_0}_\lambda\gamma_0,S^{t_0}_{\lambda_0}\gamma_0)\le m(\lambda)\to 0 \tag{13}$$

as $\lambda\to\lambda_0$ and, consequently,

$$d(S^{t_0}_{\lambda_k}\gamma_k,S^{t_0}_{\lambda_0}\gamma_0)\le d(S^{t_0}_{\lambda_k}\gamma_k,S^{t_0}_{\lambda_k}\gamma_0)+d(S^{t_0}_{\lambda_k}\gamma_0,S^{t_0}_{\lambda_0}\gamma_0)\le$$
$$\mathcal{N}e^{-\nu t_0}d(\gamma_k,\gamma_0)+m(\lambda_k)\to 0$$

as $\lambda_k\to\lambda_0$. By Lemma 1 (see also Remark 1) for each $\lambda\in\Lambda$ the semi-group $\{S^t_\lambda\}_{t\in\mathbb{T}}$ admits a unique stationary point $\gamma_\lambda\in\Gamma(Y,X)$ and the mapping $\lambda\to\gamma_\lambda$ is continuous.

Let us write by $K_\lambda:=\gamma_\lambda(Y)$, then K_λ is a nonempty compact and invariant set of the dynamical system $(X,\mathbb{T}_1,\pi_\lambda)$. From the inequality (3) it follows that

$$\lim_{t\to+\infty}\rho(\pi^t_\lambda M,K)=0$$

for all $M\in K(X)$ and, consequently, the dynamical system $(X,\mathbb{T}_1,\pi_\lambda)$ is compactly dissipative and its Levinson center $J_\lambda\subseteq K_\lambda$. On the other hand, $K_\lambda\subseteq J_\lambda$, because the set $K_\lambda=\gamma_\lambda(Y)$ is compact and invariant, but J_λ is the maximal compact invariant set of $(X,\mathbb{T}_1,\pi_\lambda)$. Thus we have $J_\lambda=\gamma_\lambda(Y)$.

Now let $\mathbb{T}_2=\mathbb{S}$. Then we will show that the set $\gamma_\lambda(y)$ contains a single point for any $y\in Y$. If we suppose that it is not true, then there are $y_0\in Y$ and $x_1,x_2\in\gamma_\lambda(y_0)$ $(x_1\ne x_2)$. Let $\phi_i\in\Phi_{x_i}$ $(i=1,2)$ be such that $\phi_i(\mathbb{S})\subseteq J_\lambda$. Then we have

$$\pi^t_\lambda(\phi_i(-t))=x_i\ (i=1,2) \tag{14}$$

for all $t\in\mathbb{T}_1$. Note that from inequality (3) and equality (14) it follows that

$$\rho(x_1,x_2)=\rho(\pi^t_\lambda(\phi_1(-t)),\pi^t_\lambda(\phi_2(-t)))\le$$
$$Ne^{-\nu t}\rho(\phi_1(-t),\phi_2(-t))\le Ne^{-\nu t}C \tag{15}$$

for all $t \in \mathbb{T}$, where $C := \sup\{\rho(\phi_1(s), \phi_2(s)) \; : \; s \in \mathbb{S}\}$. Passing to the limit in (15) as $t \to +\infty$ we obtain $x_1 = x_2$. The obtained contradiction proves our statement.

Thus, if $\mathbb{T}_2 = \mathbb{S}$, the unique fix point $\gamma_\lambda \in \Gamma(Y, X)$ of the semi-group of operators $\{S^t_\lambda\}_{t\in\mathbb{T}_1}$ is a single-valued function and, consequently, it is continuous. Finally, inequality (4) follows from (3), because $h(\gamma_\lambda(h(x))) = (h \circ \gamma_\lambda)(h(x)) = h(x)$ for all $x \in X$. □

Remark 4. If $(Y, \mathbb{T}_2, \sigma)$ is a semi-group dynamical system (i.e. $\mathbb{T}_2 = \mathbb{R}_+$ or $\mathbb{Z}_+$), then the unique invariant section γ_λ of the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi_\lambda), \; (Y, \mathbb{T}_2, \; \sigma), h\rangle$ is multi-valued (i.e. $\gamma_\lambda(y)$ contains, generally speaking, more than one point). This fact is confirmed by the below example, which is a slight modification of example from [25, Ch1,p.42-43].

Example 1. Let $Y := [-1, 1]$ and $(Y, \mathbb{Z}_+, \sigma)$ be a cascade generated by positive powers of the odd function g, defined on $[0, 1]$ in the following way:

$$g(y) = \begin{cases} -2y & , \quad 0 \le y \le \frac{1}{2} \\ 2(y-1) & , \quad \frac{1}{2} < y \le 1. \end{cases}$$

It is easy to check that $g(Y) = Y$. Let us put $X := \mathbb{R} \times Y$ and denote by $(X, \mathbb{Z}_+, \pi)$ a cascade generated by the positive powers of the mapping $P : X \to X$

$$P \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} f(u, y) \\ g(y) \end{pmatrix}, \tag{16}$$

where $f(u, y) := \frac{1}{10}u + \frac{1}{2}y$. Finally, let $h = pr_2 : X \to Y$. From (16), it follows that h is a homomorphism of $(X, \mathbb{Z}_+, \pi)$ onto $(Y, \mathbb{Z}_+, \sigma)$ and, consequently, $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$ is a non-autonomous dynamical system. Note that

$$|(u_1, y) - (u_2, y)| = |u_1 - u_2| = 10|P(u_1, y) - P(u_2, y)|. \tag{17}$$

From (17), it follows that

$$|P^n(u_1, y) - P^n(u_2, y)| \le \mathcal{N} e^{-\nu n} |\langle u_1, y\rangle - \langle u_2, y\rangle| \tag{18}$$

for all $n \in \mathbb{Z}_+$, where $\mathcal{N} = 1$ and $\nu = \ln 10$. By Theorem 3 there exists a unique β-continuous invariant section $\gamma \in \Gamma(Y, X)$ of non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}_+, \sigma), h\rangle$. According to [25, p.43] $\gamma(y)$ is homeomorphic to the Cantor set for all $y \in [-1, 1]$.

4. **Iterated function systems, discrete inclusions and cocycles.**

Definition 9. A iterated function system (IFS) consists of a complete metric space (X, ρ) together with a finite set of mappings $f_i : X \mapsto X$ $(i = 1, \dots, m)$ (the notation $\{X; \; f_i, \; i = 1, \dots, m\}$). The IFS $\{X; \; f_i, \; i = 1, \dots, m\}$ is called hyperbolic if every function f_i $(i = 1, \dots, m)$ is a contraction.

Let W be a topological space. Denote by $C(W)$ the space of all continuous operators $f : W \to W$ equipped with the compact-open topology. Consider a set of operators $\mathcal{M} \subseteq C(W)$ and, respectively, an ensemble (collage) of discrete dynamical systems $(W, f)_{f\in\mathcal{M}}$ ((W, f) is a discrete dynamical system generated by positive powers of map f).

Definition 10. A discrete inclusion $DI(\mathcal{M})$ is (see, for example, [14]) a set of all sequences $\{\{x_j\} \mid j \ge 0\} \subset W$ such that

$$x_j = f_{i_j} x_{j-1}$$

for some $f_{i_j} \in \mathcal{M}$ (trajectory of $DI(\mathcal{M})$), i.e.

$$x_j = f_{i_j} f_{i_{j-1}} ... f_{i_1} x_0 \text{ all } f_{i_k} \in \mathcal{M}.$$

Definition 11. A bilateral sequence $\{\{x_j\} \mid j \in \mathbb{Z}\} \subset W$ is called a full trajectory of $DI(\mathcal{M})$ (entire trajectory or trajectory on $\mathbb{Z}$), if $x_{n+j} = f_{i_j} x_{n+j-1}$ for all $n \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$.

Let us consider the set-valued function $F : W \to K(W)$ defined by the equality $F(x) := \{f(x) \mid f \in \mathcal{M}\}$. Note that the set $F(x)$ is compact because $\mathcal{M}$ is so. Then the discrete inclusion $DI(\mathcal{M})$ is equivalent to the difference inclusion

$$x_j \in F(x_{j-1}). \tag{19}$$

Denote by $\mathcal{F}_{x_0}$ the set of all trajectories of discrete inclusion (19) (or $DI(\mathcal{M})$) issuing from the point $x_0 \in W$ and $\mathcal{F} := \bigcup \{\mathcal{F}_{x_0} \mid x_0 \in W\}$.

Below we will give a new approach concerning the study of discrete inclusions $DI(\mathcal{M})$ (or difference inclusion (19)). Denote by $C(\mathbb{Z}_+, W)$ the space of all continuous mappings $f : \mathbb{Z}_+ \to W$ equipped with the compact-open topology. Let $(C(\mathbb{Z}_+, W), \mathbb{Z}_+, \sigma)$ be the dynamical system of translations (shift dynamical system or dynamical system of Bebutov [24, 26]) on $C(\mathbb{Z}_+, W)$, i.e. $\sigma(k, f) := f_k$ and f_k is a $k \in \mathbb{Z}_+$ shift of f (i.e. $f_k(n) := f(n+k)$ for all $n \in \mathbb{Z}_+$).

We may now rewrite equation (19) in the following way:

$$x_{j+1} = \omega(j) x_j, \ (\omega \in \Omega := C(\mathbb{Z}_+, \mathcal{M})) \tag{20}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(j) := f_{i_{j+1}}$ for all $j \in \mathbb{Z}_+$. We denote by $\varphi(n, x_0, \omega)$ the solution of equation (20) issuing from the point $x_0 \in W$ at the initial moment $n = 0$. Note that $\mathcal{F}_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and $\mathcal{F} = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in W, \omega \in \Omega\}$, i.e. $DI(\mathcal{M})$ (or inclusion (19)) is equivalent to the family of non-autonomous equations (20) ($\omega \in \Omega$).

From the general properties of difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times W \times \Omega \to W$ satisfies the following conditions:

- (i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in W \times \Omega$;
- (ii) $\varphi(n + \tau, x_0, \omega) = \varphi(n, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in W \times \Omega$;
- (iii) the mapping φ is continuous;
- (iv) for any $n, \tau \in \mathbb{Z}_+$ and $\omega_1, \omega_2 \in \Omega$ there exists $\omega_3 \in \Omega$ such that
$$U(n, \omega_2) U(\tau, \omega_1) = U(n + \tau, \omega_3), \tag{21}$$
where $\omega \in \Omega$, $U(n, \omega) := \varphi(n, \cdot, \omega) = \prod_{k=0}^{n} \omega(k)$, $\omega(k) := f_{i_k}$ $(k = 0, 1, \ldots, n)$ and $f_{i_0} := Id_W$.

Let W, Ω be two topological spaces and $(\Omega, \mathbb{T}, \sigma)$ be a semi-group dynamical system on Ω.

Definition 12. Recall [24] that a triplet $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma) \rangle$ (or briefly φ) is called a cocycle over $(\Omega, \mathbb{T}, \sigma)$ with the fiber W, if φ is a mapping from $\mathbb{T} \times W \times \Omega$ to W satisfying the following conditions:

1. $\varphi(0, x, \omega) = x$ for all $(x, \omega) \in W \times \Omega$;
2. $\varphi(n + \tau, x, \omega) = \varphi(n, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ for all $n, \tau \in \mathbb{T}$ and $(x, \omega) \in W \times \Omega$;
3. the mapping φ is continuous.

Let $X := W \times \Omega$, and define the mapping $\pi : X \times \mathbb{T} \to X$ by the equality: $\pi((u, \omega), t) := (\varphi(t, u, \omega), \sigma(t, \omega))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to check that

$(X, \mathbb{T}, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system [1], [24]; but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X, \mathbb{T}, \pi)$ onto $(\Omega, \mathbb{T}, \sigma)$ and hence $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \sigma), h \rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma) \rangle$ over the dynamical system $(\Omega, \mathbb{T}, \sigma)$ with the fiber W, then there can be constructed a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}, \sigma), h \rangle$ $(X := W \times \Omega)$, which we will call a non-autonomous dynamical system generated (associated) by the cocycle $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma) \rangle$ over $(\Omega, \mathbb{T}, \sigma)$.

From that which has been presented above, it follows that every $DI(\mathcal{M})$ (respectively, inclusion (19)) in a natural way generates a cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(n, x, \omega)$ is the solution of equation (20) issuing from the point $x \in W$ at the initial moment $n = 0$. Thus, we can study inclusion (19) (respectively, $DI(\mathcal{M})$) in the framework of the theory of cocycles with discrete time.

Theorem 4. [10] *The following statements hold:*

(i) $\Omega = \overline{Per(\sigma)}$, *where* $Per(\sigma)$ *is the set of all periodic points of* $(\Omega, \mathbb{Z}_+, \sigma)$ *(i.e.* $\omega \in Per(\sigma)$, *if there exists* $\tau \in \mathbb{N}$ *such that* $\sigma(\tau, \omega) = \omega$*);*

(ii) *the set* Ω *is compact;*

(iii) Ω *is invariant, i.e.* $\sigma^t \Omega = \Omega$ *for all* $t \in \mathbb{Z}_+$;

(iv) *if* $\mathcal{M}$ *is a compact subset of* $C(W)$ *and* $\langle W, \phi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ *is a cocycle generated by* $DI(\mathcal{M})$*, then* φ *satisfies the condition (21).*

5. **Some properties of Lipschitzian mappings.** Let (W, ρ) be a metric space.

Definition 13. A mapping $f : W \to W$ satisfies the Lipschitz condition, if there exists a constant $L > 0$ such that $\rho(f(x_1), f(x_2)) \leq L\rho(x_1, x_2)$ for all $x_1, x_2 \in W$. The smallest constant with the above mentioned property is called the Lipschitz constant $Lip(f)$ of the mapping f.

Denote by $Lip(W) := \{f : W \mapsto W \mid Lip(f) < \infty\}$.

Lemma 2. *Let* $f \in Lip(W)$*, then the following statement hold:*

(i) $f^n \in Lip(W)$ *for all* $n \in \mathbb{N}$*, where* $f^n := f^{n-1} \circ f$ $(\forall\ n \geq 2)$*;*

(ii) $Lip(f^n) \leq Lip(f)^n$ $(\forall\ n \in \mathbb{N})$*;*

(iii) *there exists the limit*

$$r_f := \lim_{n\to\infty} (Lip(f^n))^{\frac{1}{n}};$$

(iv) $r_f \leq Lip(f)$.

Proof. The first, second and fourth statements are obvious. To prove the third statement we note that the sequence $\{b_n\}$ $(b_n := \ln(Lip(f^n)))$ is sub-additive, i.e. $b_{n_1+n_2} \leq b_{n_1} + b_{n_2}$ for all $n_1, n_2 \in \mathbb{N}$. Thus there exists the limit $\lim\limits_{n\to\infty} \frac{b_n}{n}$ (see, for example, [19, p.27]) and, consequently, there exists also the limit

$$\lim_{n\to\infty} (Lip(f^n))^{\frac{1}{n}} = e^{\lim_{n\to\infty} \frac{b_n}{n}}.$$

□

Definition 14. The spectral radius of function $f \in Lip(W)$ is said to be the number $r_f := \lim\limits_{n\to\infty} (Lip(f^n))^{\frac{1}{n}}$.

Definition 15. The function $f \in Lip(W)$ is said to be a generalized contraction (contracting in the extended sense) if $r_f < 1$.

Remark 5. 1. If $f \in Lip(W)$ is a contraction (i.e., $Lip(f) < 1$), then $r_f < 1$ because $r_f \leq Lip(f)$.

2. If $f \in Lip(W)$ and $r_f < 1$ then, generally speaking, f is not a contraction. This fact is confirmed by the below example. In fact, let $W := C[0,1]$ and $f \in Lip(W)$ is defined by equality

$$(f\varphi)(t) := \frac{3}{2}\int_0^t \varphi(s)ds$$

($t \in [0,1]$ and $\varphi \in C[0,1]$). It is easy to verify that $Lip(f^n) = (\frac{3}{2})^n \frac{1}{n!}$. In particular, $Lip(f) = \frac{3}{2}, Lip(f^2) = \frac{9}{8}$ and $Lip(f^3) = \frac{27}{32}$. In addition $Lip(f^n) \leq 2(\frac{3}{4})^n$ for all $n \in \mathbb{N}$. Thus the mapping f is a generalized contraction, but $Lip(f) \geq 1$.

Lemma 3. *The function $f \in Lip(W)$ is a generalized contraction if and only if there exist positive numbers $\mathcal{N}$ and ν $(0 < \nu < 1)$ such that*

$$Lip(f^n) \leq \mathcal{N}\nu^n \tag{22}$$

for all $n \in \mathcal{N}$.

Proof. It is easy to see that from (22) we have $r_f \leq \nu < 1$.

Let now $r_f < 1$ and $\varepsilon \in (0, 1 - r_f)$. Then there is a number $n_0 = n_0(\varepsilon) \in \mathcal{N}$ such that $(Lip(f^n))^{\frac{1}{n}} < r_f + \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_0$. We put $\nu := r_f + \varepsilon$ and $\mathcal{N} := \max\{1, \nu Lip(f), \nu^2 Lip(f^2), \ldots, \nu^{n_0} Lip(f^{n_0})\}$, then $Lip(f^n) \leq \mathcal{N}\nu^n$ for all $n \in \mathbb{N}$. □

Corollary 1. *The mapping f is a generalized contraction if and only if one of its iterates is contracting.*

Definition 16. A subset of operators $\mathcal{M} \subseteq C(W)$ is said to be generally contracting (contracting in the extended sense), if there are positive numbers $\mathcal{N}$ and $\nu < 1$ such that

$$L(f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1}) \leq \mathcal{N}\nu^n$$

for all $f_{i_1}, f_{i_2}, \ldots, f_{i_n} \in \mathcal{M}$ and $n \in \mathbb{N}$.

Remark 6. 1. If the subset of operators $\mathcal{M} \subseteq C(W)$ is generally contracting, then

(i) every function $f \in \mathcal{M}$ is generally contracting;

(ii) every function $f := f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1}$ ($f_{i_k} \in \mathcal{M}$ for all $k = 1, \ldots, n$) is a generalized contraction.

2. If $r_f < 1$ for every function $f \in \mathcal{M}$,then the subset of operators $\mathcal{M} \subseteq C(W)$, generally speaking, is not a generalized contraction. In fact, let $W := \mathbb{R}^2$ and $\mathcal{M} \subseteq C(W)$ consists from two functions $\{f_1, f_2\}$, where $f_1(x_1, x_2) := (2x_2, \frac{x_1}{4})$ and $f_2(x_1, x_2) := (5x_2, \frac{x_1}{6})$. Then $r_{f_1} = \frac{\sqrt{2}}{2}$, $r_{f_2} = \sqrt{\frac{5}{6}}$ and $r_{f_1 f_2} = \frac{5}{4}$ (see [12]) and, consequently, $\mathcal{M} := \{f_1, f_2\}$ is not generally contracting.

Lemma 4. *Let $\mathcal{M} = \{f_1, f_2, \ldots, f_m\}$, then the following statements hold:*

(i) *If $Lip(f_i) < 1$ for all $1 \leq i \leq m$, then the subset of operators $\mathcal{M} \subseteq C(W)$ is generally contracting;*

(ii) *Let $r_{f_i} < 1$ for all $1 \leq i \leq m$ and the mappings $f_1, \ldots, f_m$ are permutable (i.e. $f_i \circ f_j = f_j \circ f_i$ for all $1 \leq i, j \leq m$), then the set of operators $\mathcal{M} = \{f_1, \ldots, f_m\}$ is generally contracting.*

Proof. Let $Lip(f_i) < 1$ for all $i = 1, \dots, m$. Then $Lip(f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}) \leq Lip(f_{i_n}) \dots Lip(f_{i_1}) \leq \nu^n$ for all $n \in \mathbb{N}$, where $\nu := \max\{Lip(f_k) \mid 1 \leq k \leq m\}$.

Let $n \in \mathbb{N}$ and $f_{i_k} \in \mathcal{M} := \{f_1, \dots, f_m\}$ $(1 \leq i_k \leq m$ for all $1 \leq k \leq n)$. Then $f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1} = f_1^{k_1} \dots f_m^{k_m}$, where $k_i \in \mathbb{Z}_+$ $(1 \leq i \leq m)$ with condition $k_1 + \dots + k_m = n$. Thus we have

$$Lip(f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}) = Lip(f_1^{k_1}) \dots Lip(f_m^{k_m}). \tag{23}$$

Since $r_{f_i} < 1$, then by Lemma 3 there are positive numbers $\mathcal{N}_i$ and $\nu_i < 1$ such that

$$Lip(f_i^n) \leq \mathcal{N}_i \nu_i^n \tag{24}$$

for all $n \in \mathbb{N}$.

From the relations (23) and (24), it follows that

$$Lip(f_{i_n} \circ f_{i_{n-1}} \circ \dots \circ f_{i_1}) \leq \mathcal{N} \nu^n$$

for all $n \in \mathbb{N}$, where $\mathcal{N} := \max\{\mathcal{N}_k \mid 1 \leq k \leq m\}$ and $\nu := \max\{\nu_k \mid 1 \leq k \leq m\}$. □

6. **Relation between compact global attractors of skew-product systems, collages and cocycles.**

Theorem 5. [10] *Suppose the following conditions are fulfilled:*

(i) $\mathcal{M} := \{f_i \ : i \in I\}$ *is a compact subset from* $C(W)$*;*
(ii) *the set* $\mathcal{M}$ *of operators is contracting in the extended sense.*

Then the set-valued cascade (W, F) *(discrete dynamical system generated by positive powers of mapping* F*) is compactly dissipative, , where* $F(x) := \{f(x) \mid f \in \mathcal{M}\}$ *(*$\forall x \in W$*).*

Theorem 6. [10] *Let* $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma\rangle$ *be a cocycle,* Ω *be a compact space and* $f : \mathbb{T} \times W :\to K(W)$ *be a mapping defined by the equality*

$$f(t, u) = \varphi(t, u, \Omega) \tag{25}$$

for all $u \in W$ *and* $t \in \mathbb{T}$*.*

Then the mapping f *possesses the following properties:*

a. $f(0, u) = u$ *for all* $u \in W$*;*
b. $f(t, f(\tau, u)) \subseteq f(t + \tau, u)$ *for all* $t, \tau \in \mathbb{T}$ *and* $u \in W$*;*
c. $f : \mathbb{T} \times W \to K(W)$ *is upper semi-continuous, i.e.*

$$\lim_{t \to t_0, u \to u_0} \beta(f(t, u), f(t_0, u_0)) = 0 \quad \forall (t_0, u_0) \in \mathbb{T} \times W;$$

d. *if the cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ *satisfies the following condition:*

$$\forall t, \tau \in \mathbb{T}, u_1, u_2 \in W \ \exists u_3 \text{ such that } \varphi(t, \varphi(\tau, x, u_1), u_2) = \varphi(t + \tau, x, u_3), \tag{26}$$

then

$$f(t, f(\tau, u)) = f(t + \tau, u)$$

for all $t, \tau \in \mathbb{T}$ *and* $u \in W$*.*

Corollary 2. *Every cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma\rangle$ *with the compact* Ω *and satisfying the condition (26) generates a set-valued dynamical system* $(W, \mathbb{T}, f)$*, where* $f : \mathbb{T} \times W \to K(W)$ *is defined by equality (25).*

Definition 17. A cocycle φ over $(\Omega, \mathbb{T}, \sigma)$ with the fiber W is said to be a compactly dissipative one, if there is a nonempty compact $K \subseteq W$ such that

$$\lim_{t \to +\infty} \sup\{\beta(U(t,\omega)M, K) \mid \omega \in \Omega\} = 0 \tag{27}$$

for any $M \in K(W)$, where $U(t,\omega) := \varphi(t,\cdot,\omega)$.

Definition 18. [7, Ch.II] A metric space X possesses the property (S), if for every compact subset $K \subseteq X$ there exists a connected compact subset $L \subseteq X$ such that $K \subseteq L$.

Theorem 7. [7, Ch.II] *Let Y be compact, $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (27). Then the following statements hold:*

- (i) *$w \in I_y$ $(y \in Y)$ if and only if there exits a complete trajectory $\nu : \mathbb{S} \to W$ of the cocycle φ, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{S})$ is relatively compact;*
- (ii) *I_y $(y \in Y)$ is connected, if the space W possesses the property (S).*

Definition 19. The smallest compact set $I \subseteq W$ with property (27) is said to be a Levinson center of cocycle φ.

Theorem 8. [10]

- (i) *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ be a cocycle with the compact Ω and satisfying the condition (26). Then the following statements are equivalent:*
  - (a) *the cocycle φ is compactly dissipative;*
  - (b) *the skew-product dynamical system $(X, \mathbb{T}, \pi)$ generated by the cocycle φ is compactly dissipative;*
  - (c) *the set-valued dynamical system $(W, \mathbb{T}, f)$ generated by the cocycle φ is compactly dissipative.*
- (ii) *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ be a compact dissipative cocycle and the following conditions be fulfilled:*
  - (a) *Ω is compact and invariant ($\sigma^t\Omega = \Omega$ for all $t \in \mathbb{T}$);*
  - (b) *the cocycle φ satisfies condition (26).*

  Then $I = pr_1(J)$, where J is the Levinson's center of the skew-product dynamical system $(X, \mathbb{T}, \pi)$ (generated by the cocycle φ) and I is the Levinson center of the set-valued dynamical system $(W, \mathbb{T}, f)$ (generated by the cocycle φ).

Denote by $\Phi(\varphi)$ the set of all full trajectories of the cocycle φ.

Corollary 3. *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma)\rangle$ be a compactly dissipative cocycle and the following conditions be fulfilled:*

- (i) *Ω is compact and invariant;*
- (ii) *the cocycle φ satisfies condition (26).*

Then $I = \{u \in W \;:\; \exists \eta \in \Phi(\varphi),\ \eta(0) = u$ and $\eta(\mathbb{S})$ is relatively compact$\}$.

7. **Continuous dependence of attractors of IFS.**

Theorem 9. [10] *Suppose that the following conditions are fulfilled:*

- (i) *$\mathcal{M}$ is a compact subset of $C(W)$;*
- (ii) *$\mathcal{M}$ is contracting in the extended sense.*

Then

(i) $I_\omega := \{u \in W \ : \ $ *a solution* $\varphi(n,u,\omega)$ *of equation (20) is defined on* $\mathbb{Z}$ *and* $\varphi(\mathbb{Z},u,\omega)$ *is relatively compact*$\} \neq \emptyset$ *for all* $\omega \in \Omega$, *i.e. every equation (20) admits at least one solution defined on* $\mathbb{Z}$ *with relatively compact range of values;*

(ii) *the sets* I_ω $(\omega \in \Omega)$ *and* $I := \bigcup\{I_\omega \ : \ \omega \in \Omega\}$ *are compact;*

(iii) *the set-valued map* $\omega \to I_\omega$ *is upper semi-continuous;*

(iv) *the family of compact sets* $\{I_\omega \ : \ \omega \in \Omega\}$ *is invariant with respect to the cocycle* φ, *i.e.* $\varphi(n,I_\omega,\omega) = I_{\sigma^n \omega}$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$;

(v) $\rho(\varphi(n,u_1,\omega),\varphi(n,u_2,\omega)) \le \mathcal{N} e^{-\nu n}\rho(u_1,u_2)$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$ *and* $u_1,u_2 \in W$, *where* $\mathcal{N}$ *and* ν *are positive numbers from the definition of the contractivity of* $\mathcal{M}$ *in the extended sense;*

(vi) *if every map* $f \in \mathcal{M}$ *is invertible, then*

  (a) I_ω *consists of a single point* u_ω;

  (b) *the map* $\omega \to u_\omega$ *is continuous;*

  (c) $\varphi(n,u_\omega,\omega) = u_{\sigma(n,\omega)}$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$;

  (d) $\rho(\varphi(n,u,\omega),\varphi(n,u_\omega,\omega)) \le \mathcal{N} e^{-\nu n}\rho(u,u_\omega)$ *for all* $n \in \mathbb{Z}_+$ *and* $\omega \in \Omega$.

Let Λ be a compact metric space. Denote by $C(\Lambda \times W, W)$ the space of all continuous functions $f : \Lambda \times W \mapsto W$ equipped with compact-open topology. If $f \in C(\Lambda \times W, W)$ then we denote by $f^\lambda := f(\lambda,\cdot) \in C(W)$ and $\mathcal{M}^\lambda := \{f^\lambda \mid f \in \mathcal{M}\}$.

Consider a set of operators $\mathcal{M} \subseteq C(\Lambda \times W, W)$ and, respectively, an ensemble (collage) of discrete dynamical systems $(W, f_\lambda)_{f_\lambda \in \mathcal{M}_\lambda}$ ((W,f_λ) is a discrete dynamical system generated by positive powers of map f_λ).

We consider the equation

$$x_{j+1} = \omega(\cdot,j)x_j, \ (\omega \in \Omega := C(\Lambda \times \mathbb{Z}_+, \mathcal{M})) \tag{28}$$

or

$$x_{j+1} = \omega(\lambda,j)x_j, \ (\lambda \in \Lambda, \ \omega(\lambda,\cdot) \in \Omega_\lambda := C(\mathbb{Z}_+, \mathcal{M})), \tag{29}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(\cdot,j) := f_{i_{j+1}} \in C(\Lambda \times W, W)$ (or $\omega(\lambda,j) := f^\lambda_{i_{j+1}} \in C(W,W)$ for all $\lambda \in \Lambda$) for all $j \in \mathbb{Z}_+$, i.e. $\omega(j)$ is a continuous function depending on two variables $\lambda \in \Lambda$ and $x \in W$. We denote by $\varphi(\cdot,n,x_0,\omega)$ the solution of equation (28) (respectively, by $\varphi(\lambda,n,x_0,\omega)$ the solution of equation (29)) issuing from the point $x_0 \in W$ at the initial moment $n = 0$.

From the general properties of difference equations it follows that the mapping $\varphi : \Lambda \times \mathbb{Z}_+ \times W \times \Omega \to W$ satisfies the following conditions:

(i) $\varphi(\lambda,0,x_0,\omega) = x_0$ for all $(\lambda,x_0,\omega) \in \Lambda \times W \times \Omega$;

(ii) $\varphi(\lambda,n+\tau,x_0,\omega) = \varphi(\lambda,n,\varphi(\lambda,\tau,x_0,\omega),\sigma(\tau,\omega))$ for all $n,\tau \in \mathbb{Z}_+$ and $(\lambda,x_0,\omega) \in \Lambda \times W \times \Omega$;

(iii) the mapping φ is continuous;

(iv) for any $n,\tau \in \mathbb{Z}_+$ and $\omega_1,\omega_2 \in \Omega$ there exists $\omega_3 \in \Omega$ such that

$$U(\lambda,n,\omega_2)U(\lambda,\tau,\omega_1) = U(\lambda,n+\tau,\omega_3),$$

where $\omega \in \Omega$, $U(\lambda,n,\omega) := \varphi(\lambda,n,\cdot,\omega) = \prod_{k=0}^n \omega(\lambda,k)$, $\omega(\lambda,k) := f^\lambda_{i_k}$ $(k = 0,1,\dots,n)$ and $f^\lambda_{i_0} := Id_W$.

Let $X := W \times \Omega$, and define the mapping $\pi_\lambda : X \times \mathbb{T} \to X$ by the equality: $\pi_\lambda((u,\omega),t) := (\varphi(\lambda,t,u,\omega),\sigma(t,\omega))$ (i.e. $\pi_\lambda = (\varphi_\lambda,\sigma)$). Then it is easy to check that for each $\lambda \in \Lambda$ the triplet $(X,\mathbb{T},\pi_\lambda)$ is a dynamical system on X, but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X,\mathbb{T},\pi_\lambda)$ onto $(\Omega,\mathbb{T},\sigma)$ and hence

$\langle (X, \mathbb{T}, \pi_\lambda),\ (\Omega, \mathbb{T}, \sigma), h\rangle$ is a family of non-autonomous dynamical systems depending on parameter $\lambda \in \Lambda$. Applying Theorem 3 to the family of dynamical systems $\langle (X, \mathbb{T}, \pi_\lambda),\ (\Omega, \mathbb{T}, \sigma), h\rangle$ we will receive the following result.

Theorem 10. *Suppose that the following conditions hold:*

(i) *Λ be a compact metric space;*

(ii) *$\mathcal{M}$ be a nonempty compact subset of $C(\Lambda \times W, W)$, where W is a complete metric space;*

(iii) *the subset $\mathcal{M} \subseteq C(\Lambda \times W, W)$ is generalized contracting, i.e. there are two positive numbers $\mathcal{N}$ and $\nu < 1$ such that $Lip(f^\lambda_{i_n} \circ \ldots \circ f^\lambda_{i_1}) \le \mathcal{N}\nu^n$ for all $\lambda \in \Lambda$, $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{N}$ where $f^\lambda_k := f_k(\lambda, \cdot)$ and $f_k \in \mathcal{M}$.*

Then the following statements hold:

(i) *for each $\lambda \in \Lambda$ the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi_\lambda),\ (\Omega,\ \mathbb{Z}_+,\ \sigma),\ h\rangle$ is compactly dissipative;*

(ii)
$$\rho(\pi_\lambda(n, x_1), \pi_\lambda(n, x_2)) \le \mathcal{N}\nu^n \rho(x_1, x_2) \tag{30}$$
for all $n \in \mathbb{Z}_+$ and $x_,x_2 \in X$ $(h(x_1) = h(x_2))$, i.e. the family of non-autonomous dynamical systems $\langle (X, \mathbb{Z}_+, \pi_\lambda), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is generalized contracting;

(iii) *for each $(\lambda, \omega) \in \Lambda \times \Omega$ the set $I^\lambda_\omega := \{u \in W \mid$ the solution $\varphi(\lambda, n, u, \omega)$ of equation (29) defined on $\mathbb{Z}$ with relatively compact range of values $\varphi(\lambda,\ \mathbb{Z}, u, \omega)\}$ is nonempty and compact;*

(iv) *for each $\lambda \in \Lambda$ the family of subsets $\mathcal{I}^\lambda := \{I^\lambda_\omega \mid \omega \in \Omega\}$ is invariant with respect to cocycle $\varphi_\lambda := \varphi(\lambda, \cdot, \cdot, \cdot)$, i.e. $\varphi_\lambda(t, I^\lambda_\omega, \omega) = I^\lambda_{\sigma(t,\omega)}$ for all $t \in \mathbb{Z}_+$ and $\omega \in \Omega$;*

(v) *$I^\lambda_\omega = pr_1(J^\lambda_\omega)$ for all $\lambda \in \Lambda$ and $\omega \in \Omega$, where J^λ is the Levinson center of dynamical system $(X, \mathbb{Z}_+, \pi_\lambda)$;*

(vi) *for each $\lambda \in \Lambda$ the set $\mathbb{I}^\lambda := \cup\{I^\lambda_\omega \mid \omega \in \Omega\} = pr_1(J^\lambda)$ and, consequently, it is compact;*

(vii)
$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \alpha(I^\lambda_\omega, I^{\lambda_0}_\omega) = 0 \tag{31}$$
and, consequently, we have also
$$\lim_{\lambda \to \lambda_0} \alpha(\mathbb{I}^\lambda, \mathbb{I}^{\lambda_0}) = 0. \tag{32}$$

Proof. Let φ_λ be the cocycle generated by equation (29). Denote by $(X, \mathbb{Z}_+, \pi_\lambda)$ the skew-product dynamical system generated by cocycle φ_λ (i.e. $X := W \times \Omega$ and $\pi_\lambda := (\varphi_\lambda, \sigma)$). Let $\langle (X, \mathbb{Z}_+, \pi_\lambda), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ be the non-autonomous dynamical system associated by cocycle φ_λ, where $h := pr_2 : X \mapsto \Omega$. Under the conditions of Theorem the family of non-autonomous dynamical systems $\langle (X, \mathbb{Z}_+, \pi_\lambda), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ satisfies the inequality (30) because $\pi_\lambda(n, x) = (\varphi_\lambda(n, u,\ \omega),\ \sigma(n,\ \omega))$ $(x := (u, \omega))$ and $\varphi_\lambda(n, u, \omega) = \omega(\lambda, n) \circ \ldots \circ \omega(\lambda, 1)u$. By Theorem 3 for each $\lambda \in \Lambda$ dynamical system $(X, \mathbb{Z}_+, \pi_\lambda)$ admits a compact global attractor J^λ and there exists the unique invariant section $\gamma_\lambda \in \Gamma(\Omega, X)$ such that:

(i) the mapping $\lambda \mapsto \gamma_\lambda$ is continuous, i.e.
$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \alpha(\gamma_\lambda(\omega), \gamma_{\lambda_0}(\omega)) = 0; \tag{33}$$

(ii) $J^{\lambda}_{\omega} = \gamma_{\lambda}(\omega)$ for all $\omega \in \Omega$ and, consequently, $J^{\lambda} = \gamma_{\lambda}(\Omega)$,where $J^{\lambda}_{\omega} := X_{\omega} \cap J^{\lambda}$ and $X_{\omega} := h^{-1}(\omega)$.

Since $(X, \mathbb{Z}_+, \pi_{\lambda})$ is a skew-product dynamical system and $X = W \times \Omega$, then γ_{λ} has the form $(\phi_{\lambda}, Id_{\Omega})$, where $\phi_{\lambda} \in C(\Omega, W)$. Note that $I^{\lambda}_{\omega} = pr_1(J^{\lambda}_{\omega})$ and, consequently, it is non-empty and compact. On the other hand $\pi_{\lambda}(n, J^{\lambda}_{\omega}) = J^{\lambda}_{\sigma(n,\omega)}$ for all $\lambda \in \Lambda$, $n \in \mathbb{Z}_+$ and $\omega \in \Omega$ because Ω is invariant (i.e. $\sigma(n, \Omega) = \Omega$ for all $n \in \mathbb{Z}_+$) and, consequently, $\varphi_{\lambda}(n, I^{\lambda}_{\omega}, \omega) = \phi_{\lambda}(\pi_{\lambda}(n, J^{\lambda}) = \phi_{\lambda}(J^{\lambda}_{\sigma(n,\omega)}) = I^{\lambda}_{\sigma(n,\omega)}$.

From the equalities (33) and $\gamma_{\lambda} = (\phi_{\lambda}, Id_{\Omega})$ follow the equalities (31) and (32). □

Remark 7. If $\mathcal{M} \subseteq C(\Lambda \times W, W)$ is a finite set, i.e. $\mathcal{M} = \{f_1, \dots, f_m\}$, then the equality (32) coincides with Bransley's theorem of continuous dependence of fractals on parameters [2, Th.1,Ch.III] (see also [18]).

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