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IPSE: An Individualized Digital Environment for Strategic Planning at the University Level

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Abstract
This study focuses on the design and the implementation of a digital environment aimed at fostering strategic planning competence in problem-solving through individualization features: the Individualized Planned Strategy Environment (IPSE). Within IPSE, students are engaged in a sequence of oriented activities, guiding them in constructing and following a theoretically justified plan for solving a mathematical problem, thus promoting a gradual integration between conceptual and procedural knowledge. IPSE envisages also meta-level activities, aimed at fostering the handling of multiple representations toward a unifying and structural view of the subject at stake. We discuss the results of a case study conducted with engineering freshmen at the University of Salerno, involved in problem-solving activities devoted to peer assessment. This led us to identify certain student profiles both theory- and data-driven, according to the students' progress in using the components of Habermas' rationality when solving a problem. We highlighted that some students show a full realization of the dynamic nature of Habermas' model of rationality, where knowing, acting and communicating interact and intertwine.

Keywords Digital environment · Individualization · Strategic planning · Task design · University level

Introduction

In daily life, as well as in specific domains like research, marketing or learning, thinking strategically is a key competence. People's behaviour is judged rational if, given a specific goal, they construct and follow a suitable plan to achieve it;

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moreover, the plan is more likely to be effective the more it connects the goal that one is aiming towards with information at disposal, and it activates past experiences and skills acquired. So, strategic planning, as a component of strategic thinking, is a person-dependent activity: even if two people have the same goal, they could choose different plans to reach them, taking advantage of their individual resources (Pellerey, 2004; Goldman et al., 2015). In education, thinking strategically is an essential goal, mainly for problem-solving, also taking into account individual habits, attitudes and approaches. Literature shows wide debate on the possibility of teaching problem-solving strategies (Polya, 1945; Silver, 1985; Lester & Cai, 2016).

According to Schoenfeld (1980), "we cannot rely on students' abilities to grasp useful problem-solving strategies when the students are not given explicit instructions on their use and [...] the instruction 'made a difference'" (p. 798). Indeed, the teacher should offer a model of strategic thinking, that is, rational behaviour and discourse, consisting not only of facts but also their justifications (Boero & Planas, 2014), and aim to activate the students' individual resources. Unfortunately, the general conditions of some university contexts (large number of students, heterogeneous social and mathematical backgrounds, etc.) hinder teaching tailored to individual needs. So, our research wants to contribute to going forward in the stream of some recent studies investigating the possible affordances of using technology to promote students' problem-solving capabilities and flexibility (Jacinto, 2023; del Olmo-Muñoz et al., 2022; Santos-Trigo, 2019). In this respect, we designed digital individualized resources involving students in problem solving activities. We exploit technology as a non-human tutor in supporting students' awareness and capabilities in problem-solving processes, by the development and execution of a theoretically justified plan. The design takes into account Polya's steps in problem-solving (Polya, 1945): (1) understanding the problem; (2) devising a plan; (3) carrying out the plan; and (4) looking back. By regarding the solving problem process developed by the students as a progressively more rational discourse to convince themselves and others, we rely on Habermas' model of rationality (Habermas, 1999), according to its interpretation in Mathematics Education (Morselli & Boero, 2009). This theoretical lens underlies either the design of the digital resources or the analysis of the students' productions.

The hypothesis grounding the present study is that digital resources finely designed and oriented to individualized teaching can foster the development of strategic planning competence and awareness in problem-solving. The adopted theoretical framework will allow us to operationalize the concept of "strategic planning". In this respect, we present the design and implementation of an environment suitable for nurturing such kind of competence and awareness, named the Individualized Planned Strategy Environment (IPSE). As flavoured by the name, it is strongly oriented towards individuals' needs, approaches and habits and leaves the individual student a broad degree of freedom in making choices and selecting different methods and techniques to solve a problem. With respect to our research hypothesis, we discuss the outcomes of an explanatory case study (Cohen et al., 2007, p. 255) carried out with engineering students of the University of Salerno. In particular, we analyse, by means of Habermas' model of rationality, students' problem-solving processes devoted to the peer assessment before and after their interaction with IPSE. This allows us to follow how the students' productions change and make hypotheses on the impact of IPSE.

Conceptual Background

In this section, we describe the essential theoretical principles underlying the design and analysis of the digital environment IPSE.

Habermas' Construct of Rational Behaviour

Habermas' construct of rational behaviour has been used in Mathematics Education, especially as a lens for analysing the activity of proving (Habermas, 1999; Morselli & Boero, 2009). This model envisages three components of rational behaviour: an *epistemic component*, concerning the control of the propositions and their reciprocal links; a *teleological component*, concerning conscious choices to achieve the goal of the activity; and a *communicative component*, concerning the choices of suitable means of communication within a specific community. These components correspond to specific activities and behaviours in Mathematics: the epistemic component gives rise to the validation of statements based on shared premises and correct ways of reasoning; the *teleological component* is related to the problem-solving character of a task and to the choice of the better strategy to solve it; finally, the *communicative* component regards the effective sharing of steps of reasoning. We choose this framework since we regard the students' development of the solving process as a rational discourse, oriented by a specific goal and justified by some knowledge, that they carry on and communicate within their specific community (to the peers and the teacher) to convince themselves and others, hence to "ascertain" and "persuading" (Harel & Sowder, 1998). Therefore, in this view, strategic planning seems to be mainly linked to the teleological component of Habermas rational behaviour, as it foresees the student to elaborate a strategy with respect to a goal. It is also connected to the epistemic component, as the strategy should be grounded on shared knowledge, as well as to the communicative component, because of the interaction within a community (peers and teacher).

Personalization and Individualization

At the university level, students experience not only cognitive difficulties with Mathematics, but also sociocultural and psychological difficulties related to the change of environment and the transition from a close and personal relationship with their teacher and classmates to a formal and less direct one (Gueudet et al., 2016, Hochmuth et al., 2021; Di Martino et al., 2023; Telloni, 2024). It suggests the need to offer teaching that supports each student in her own learning needs and follows her progress. Recent research papers highlighted the crucial role that e-learning environments can have in supporting educational processes (Trgalová & Tabach, 2023) and, more specifically, in increasing the individualization/personalization of teaching and learning at the university level (Alessio et al., 2019; Cusi & Telloni, 2019; Engelbrecht & Borba, 2024; Lepellere et al., 2019; Silverman & Hoyos, 2018).

These environments can support teaching tailored to the learning needs of each student from different points of view. They facilitate *communicative* aspects of the didactical processes, allowing teachers to keep track of students' work and mistakes and providing ways to share resources, feedback and comments; from a *cognitive* point of view, they foster students' understanding, supporting the coordination of different semiotic systems (Duval, 2006), stimulating each learner through various learning channels and allowing visualization and explorations through online activities; at the *affective* level, they offer immersive experiences and opportunities of engagement to the students; finally, considering *assessment*, they simplify certain processes, allowing automatic or semi-automatic evaluation and (self-)assessment. In this way, students' psychological, sociocultural and cognitive difficulties can be minimized, even within the heterogeneity of large classes.

In this article, we intend the *individualization* of teaching and learning as the differentiation of the learning paths to enable students to reach common objectives (Shemshack & Spector, 2020). Individualization differs from *personalization*, which envisages the differentiation of both the learning paths and the formative goals. We chose to focus on individualization rather than on personalization for two main reasons: First, our context was a course of Mathematics for Engineering, where students should be brought to achieve minimum standards of learning so that they are able to face subsequent specialistic subjects; second, we intended to support especially the learning needs of low achieving students.

Procedural and Conceptual Understanding in Linear Algebra

A common trend in Mathematics Education distinguishes and often opposes procedural and conceptual knowledge (Donevska-Todorova, 2016; Hiebert & Lefevre, 1986; Rittle-Johnson & Schneider, 2015; Skemp, 1976), identifying, in general, the former as operative and mechanical knowledge and the latter as knowledge of higher level. On the other hand, some authors (Kieran, 2013; Maciejewski & Star, 2016; Sfard, 1991; Star, 2005) emphasize that these two kinds of conceptions of mathematics are not mutually exclusive, but complementary.

Linear Algebra is characterized by an intertwinement of procedural and conceptual understanding (Donevska-Todorova, 2016, 2018; Dorier & Sierpinska, 2001), since students should combine a structural vision of the mathematical objects and handle procedures and algorithms to transform them. More specifically, the learning of Linear Algebra is highly demanding from a cognitive point of view: on the one hand, the student needs to work with certain objects, like vectors, and transformations on them; on the other hand, both objects and transformations need to be seen as elements of structures. This requires thinking at a trans-object level (Dorier & Sierpinska, 2001), that is, according to a unifying and structural perspective. At the same time, procedural knowledge is necessary to manipulate the structures and their objects in a meaningful way.

The teaching of Linear Algebra, briefly, seems to be characterized by the following specific features: 1) three levels of languages and ways of thinking that should be integrated and coordinated to achieve meaningful learning of the subject; some authors named them “abstract”, “algebraic” and “geometric” (Donevska-Todorova, 2018; Dorier & Sierpinska, 2001); 2) many systems of semiotic representations (Duval, 2006) and the need to distinguish between a mathematical object and its representations, to switch between different representations and mainly to choose the most convenient representation in light of a specific aim – in this respect, Stewart et al. (2019) in their review highlight the need of studies which give insights on how effectively coordinate geometric and algebraic understanding; 3) need to coordinate between a local perspective and a global, generalizing and unifying one; need of cognitive flexibility – in this respect, Dogan (2019) points out on the need for students to have the flexibility to activate non-routinized plans.

The Design of IPSE

In this section, we are going to refine the research goal in terms of specific research questions:

(RQ1) How to design digital tasks supporting students’ individualized development of a strategic planning approach in mathematical problem-solving?
(RQ2) Can we observe improvement in students’ problem solving process in terms of construction and implementation of a theoretically justified plan for solving a mathematical problem and in its communication?

In the following, we describe the design of the digital environment IPSE (Individualized Planned Strategy Environment), aimed at favouring the students’ development of the capability of strategic planning toward a given goal through individualization features. IPSE consists of a route of activities, constructed according to the idea of “*divide et impera*”. It envisages the decomposition of the problem through subsequent refinements that students are required to recognize, tidying it up and deepening it. Hence, for a given problem, we identify the following elements: *sub-problems*, that is intermediate targets, the achievement of which allows solving the problem; *elementary steps*, that is goals that can be viewed also as procedures, possibly achievable in different ways and allowing to solve a sub-problem; and *procedural steps*, that is algorithms having as output the goals indicated by the elementary steps.

IPSE should lead students through a gradual, spiral-shaped transition, intentionally monitored from a metacognitive viewpoint through guiding questions, from the conceptual to the procedural level of problem-solving. Indeed, they should pass from the elaboration of the solving strategy of the problem and its justification to its execution and finally to the view of the specific faced problem as an instance of a class of problems. Therefore, the proposed definitions of sub-problems, elementary and procedural steps cannot be unambiguous since the distinction between them proceeds granularly by subsequent refinements and converts progressively goals into procedures. This implies that the distinction between these elements is context-dependent and derives from an author's choice, which can be done according to the supposed background of the students IPSE is addressed to. Each IPSE automatically provides specific and individualized feedback, designed according to the typical difficulties discussed in the literature and aiming at scaffolding the students' works. After a student receives feedback for an incorrect answer, IPSE allows her to change her answer. In other words, technology acts as a non-human tutor to support in an individualized way the student's problem-solving activity within the digital environment.

IPSE also envisages meta-level activities (Dorier et al., 2010), that is activities aimed at making explicit the significance of the specific contents for the general theory. It is expected that they induce students to reflect on the activity done and foster a unifying and structural view of the mathematical topic at stake, namely inducing switches between semiotic systems and transfers of knowledge in affine contexts. This is in tune with the actions associated with the heuristics in problem-solving identified by Lompscher (1975), i.e. *reduction*, *reversibility*, *consideration of aspects*, *change of aspects* and *transferring*.

These meta-level activities guarantee that IPSE diverges from the most advanced forms of CAI (Computer Assisted Instruction), based on behaviourist learning theory, moving towards a constructivist, metacognitive and individualized approach. Indeed, through the metacognitive activities, students are invited to reflect upon the just applied knowledge by interpreting and transferring it to other context and classes of problems. The novelty of IPSE consists in acting as an automatic scaffolding able to address individual needs and preferences, not just for learning mathematical contents, but to induce a methodological change in the students' attitude when solving a problem. In this perspective, the designer, through IPSE, fosters a progressive rationalization of the student's behaviour and discourse in problem-solving activities (Douek, 2014).

A further key feature of IPSE is its replicability, as the design principles could be applied to any mathematical content. It is worthwhile to note that individualization of learning paths has been one of the most promising features of the e-learning environments, but it seems to remain almost not fully exploited, as the lack in the recent literature can suggest. So, the present article should give a contribution to the existing literature to deepening the affordances of technology in providing individualized learning paths aimed at promoting university students' flexibility and awareness in problem solving. This should give a partial answer to the call for studies concerning the connection between flexibility, students' problem-solving capabilities and the technology acting as a non-human tutor (Jacinto, 2023; del Olmo-Muñoz et al., 2022; Santos-Trigo, 2019; Rittle-Johnson & Star, 2007).

The Rationale of the IPSE

Each IPSE is organized into five activities (Figure 1):

Activity 1: Plan. This activity aims to define the strategic planning that leads to the solution of the problem.

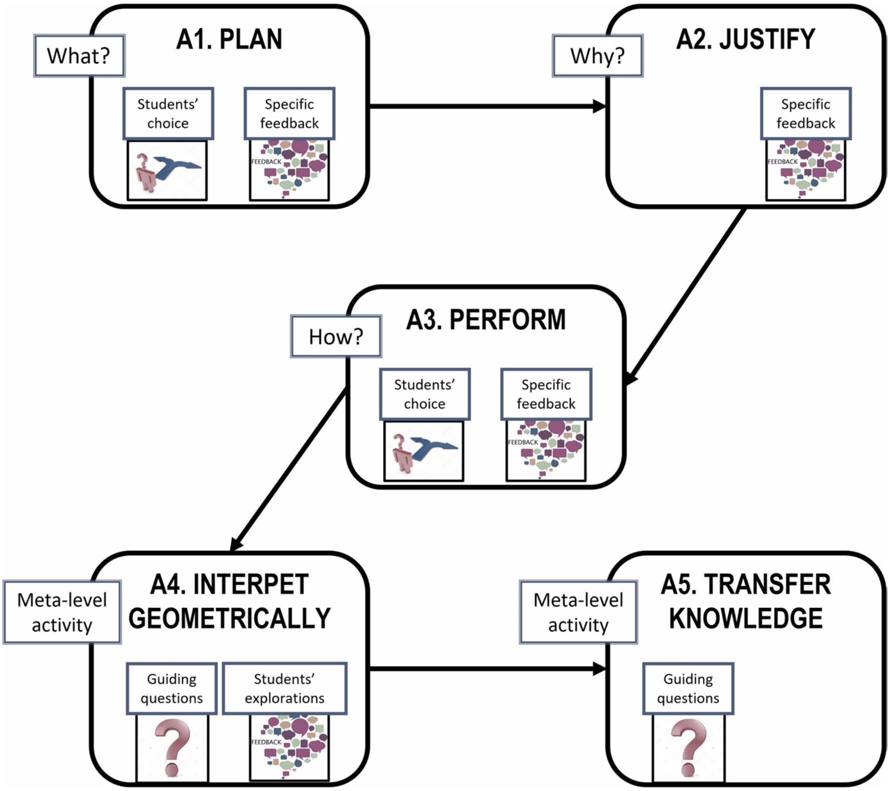


Fig. 1 The structure of IPSE

Activity 2: Justify. This activity aims to identify the theoretical results allowing the procedural steps.

Activity 3: Perform. This activity aims to implement the elementary steps through procedural steps.

Activity 4: Interpret. This activity aims to look at the problem from another point of view, e.g. to see a geometrical interpretation of the problem.

Activity 5: Transfer knowledge. This activity aims to favour the application of the knowledge grasped in the problem just faced to solve an (apparently) different problem.

Once a problem or a class of problems has been chosen, the design of IPSE envisages the following actions by the designer:

D.1) The identification of a sequence of sub-problems needed for solving the problem..

D.2) The identification of a sequence of elementary steps for each sub-problem.

D.3) The identification of theoretical results, underlying the elementary steps and to be associated with them.

D.4) The identification of various procedures for performing the identified elementary steps To capture the most common errors of the students, this last step also foresees to include some distractors.

Two further phases are provided:

D.5) The identification of a (possibly geometrical) interpretation of the problem.

D.6) The identification of another problem or class of problems that can be solved by going back to the previous one, using a suitable interpretation.

During the interaction with IPSE, according to the previous D* phases, the students are required to the following:

S.1) Tidy up the sub-problems to be addressed.

S.2) Associate each sub-problem with the elementary steps allowing to solve it.

S.3) Associate each elementary step with the theoretical results justifying it.

S.4) Perform the elementary steps through procedural steps.

S.5) Reflect on the (geometrical) interpretation of the problem.

S.6) Think deeply about other problems that can be solved analogously.

A Paradigmatic Example

To facilitate the reader, we will detail the actions by the designer of IPSE and the activities in which the student is involved following a paradigmatic example, that is the "computation of a basis of subspace spanned by a set of vectors".

Activity A1 (Plan) corresponds to the action of *reduction* of the problem (Lompscher, 1975), that is splitting the problem into its essential components.

This requires the designer, concerning D.1, the identification of the following sub-problems:

1) To determine the dimension of Span(I), say k.

2) To select appropriate k vectors of I to be a basis B of Span(I).

Concerning D.2, the designer should identify the following elementary steps associated with the subproblem 1:

1.1) To construct the matrix A having as rows (or columns) the vector in I.

1.2) To calculate the rank of A.

1.3) To set k=rk(A)=dim(Span(I)).

and the following one, associated with sub-problem 2:

2.1) To set as vectors of B the k rows (or columns) that allowed the calculation of rk(A).

On the students' side, each learner is faced with the list of the identified sub-problems (e.g. 1 and 2 above) to be solved (the Plan); moreover, the non-ordered list of the identified elementary steps (e.g. 1.1, 1.2, 1.3, 2.1 above) is also submitted to her. She is asked to tidy up the list of elementary steps, after which specific feedback is automatically provided. When the student gives the correct answer, she is required to associate each sub-problem with the corresponding elementary steps (one-to-many correspondence between sub-problems and elementary steps). Essentially, in A1 the student should identify "what" she should do to solve the problem, which corresponds to Polya's problem-solving steps "understanding the problem" and "devising a plan".

Activity A1 has been implemented as an interactive GeoGebra file, where the two mentioned lists are shown and students can drag and set blocks corresponding to elementary steps to the items of the subproblems list (see Figure 2).

Some automatic systems of checking have been planned to signal possible mistakes or notify feedback to the learner. If the student tidies up the subproblems in the wrong order, specific feedback is given, focused on the first step not correctly inserted. The feedback is immediate and facilitative; it consists mainly of stimulus questions guiding the student to solve the problem, such as "Be careful! What is the dimension of a Vector Subspace? How can you can find it?"

Activity A2 (Justify) concerns the identification of the theoretical results justifying the elementary steps pointed out in A1. This requires the designer, concerning D.3, to focus on the following theoretical results:

1) The rank of a matrix is the number of its linearly independent rows (columns).
2) The dimension of a vector space is the number of its linearly independent generators.
3) A basis of a k-dimensional vector space is formed by k linearly independent vectors of the space.

The theoretical result 1 is associated with elementary steps 1.1 and 1.2, the theoretical result 2 is associated with elementary step 1.3, and the theoretical result 3 is associated with elementary step 2.1.

In this activity, the student answers the "why" question, and she is expected to ground her choices and actions by referring to theoretical results.

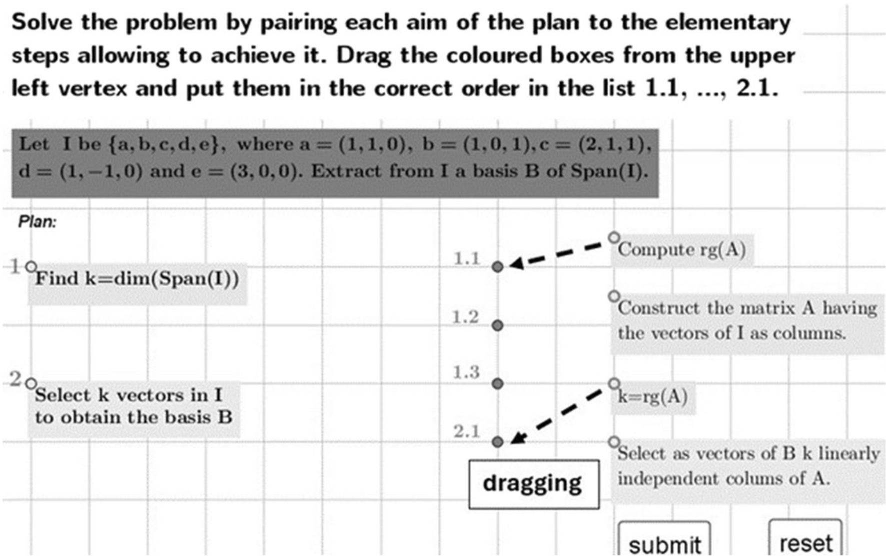


Fig. 2 Activity A1 of IPSE concerning the construction of a basis for a vector subspace

Activity A2 has also been implemented through an interactive GeoGebra file, analogous to the one described for A1, adding a new list (of theoretical results) whose items can be dragged by the students to be matched with the elementary steps.

Activity A3 (Execute) corresponds to the action of *change of aspects* of the problem (Lompscher, 1975), where the student is guided to change perspectives to find a solution. This requires the designer, concerning D.4, to identify the following procedures, some of them correct and some wrong, according to the most common difficulties afflicting the topics at stake:

1.1) The matrix A can be constructed by putting the vectors of I as rows or as columns (choice by the student).

1.2) The calculation of rk(A) can be performed by one of the following procedures (choice by the student) (Figure 3):

   a) Reduction of A to an echelon matrix.
   b) Search of a maximal non-zero minor of A (in more than one way, choice by student).
   c) Search of linearly dependent columns in A (since rk(A) must be less than or equal to 3).

For each of these possible options, some procedural calculations are provided or required, so that the student can deduce information on rk(A). The options are shown correspondingly to the student's chosen preferred procedure. For instance, in case c, the student is required to express some vectors of I as linear combinations of other vectors in I (Figure 4).

At the end of each of these steps a, b and c, the student should submit the rank of A and then the dimension *d* of Span(I). The last step of the path consists of the selection of *d* vectors in I, coherently with the calculations and the choices previously done.

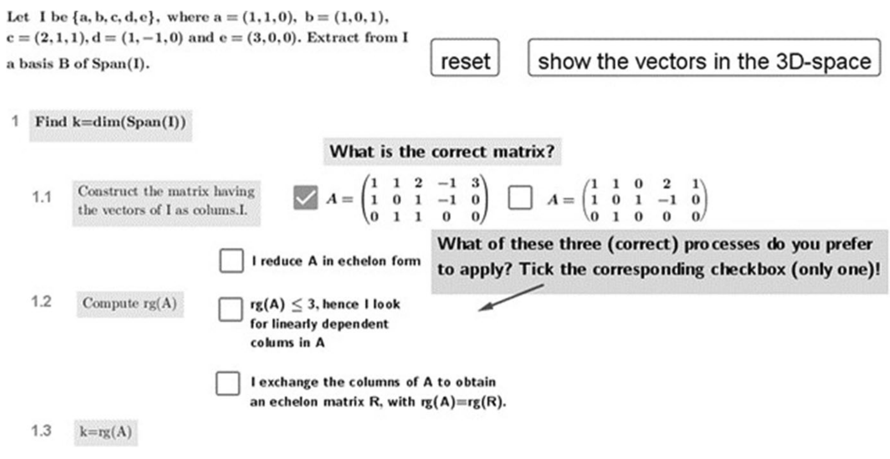


Fig. 3 Activity A3 of IPSE concerning the execution of the procedural steps

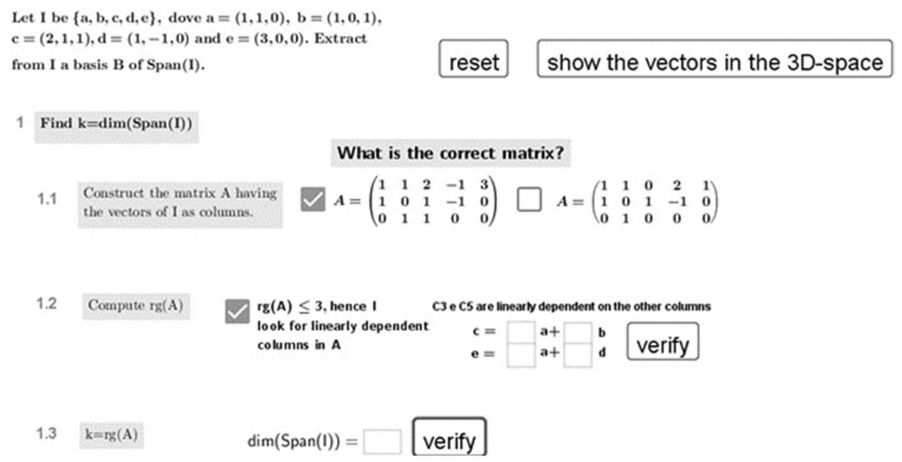


Fig. 4 Activity A3 of IPSE: justification of the linear dependence of columns in A

Activity A3 requires the student to execute procedural steps, hence addressing the question about "how" the steps can be realized, that is Polya's problem-solving step "carrying out the plan". Activity A3 has been implemented as an interactive GeoGebra file, where the choices made by the students have been carried out by control boxes or insert fields.

Activity A4 (Interpret) corresponds to the *consideration of aspects* of the problem (Lompscher, 1975), fostering the connection of knowledge, also using different semiotic registers. This requires the designer, concerning D.5, to identify a possible geometrical interpretation of the problem. In our example, it consists of the Span of 3-dimensional vectors of I in the 3-dimensional space, which can be straight lines or planes. The student has the opportunity to visualize these vectors and their Span spaces as geometrical objects and algebraic objects (their equations are shown).

Activity A5 (Transfer knowledge) corresponds to the *transferring* of the problem (Lompscher, 1975), that is moving the knowledge from one context to another one. Moreover, the *reversibility* (Lompscher, 1975) is supported, i.e. the student's moving back and forth along the thinking process. This requires the designer, concerning D.6, the identification of another problem linked to the previous one, using a suitable interpretation.

In our example, the student should find a basis for a vector subspace generated by five polynomials in $\mathbb{R}_2[t]$. Some stimulus questions and remarks guide the student to understand that this problem can be solved exactly as the previous one because of the isomorphism between $\mathbb{R}^n[t]$ and $\mathbb{R}^{n+1}$ defined by $\varphi(a_0+a_1x+...+a_nx_n)=(a_0, a_1, \ldots, a_n)$. On the students' side, the achievement of the trans-object level of thinking (Dorier & Sierpinska, 2001) is fostered by regarding different objects as elements of analogous structures, toward a unifying vision of the mathematical content at stake. Both activities A4 and A5 are meta-level activities, linked with Polya's problem-solving step "looking back". They are implemented as interactive GeoGebra files, where the student can operate on both the Graphics View and Algebra View.

Each of the previous activities has been designed according to Habermas' model of rationality. In Activities 1 and 3, the teleological component of rationality is prevalent, since the focus is on the actions and goals; in Activity 2, the epistemic component is dominant, since providing justification requires focusing on knowledge at play. The communicative component is not explicitly addressed by any Activity nor required, but correct mathematical communication is offered as a model in each Activity. As for the meta-level activities, in Activity 4, the epistemic component of rationality is prevalent since it allows the student to see different aspects of the same mathematical topic; in Activity 5, the teleological component of rationality comes into play, mainly for its feature of being "a projection from the past to the future" and consciously enriching the sets of strategies towards future problems (Boero & Planas, 2014).

Individualization Features in the IPSE

Within IPSE, the individualization of teaching and learning is realized through various modalities: First, the student is required to make choices during the interaction in the digital environment, so she is free to follow her preferences, according to both the conceptual and the procedural perspectives. Indeed, the IPSE is designed so that different correct resolution strategies are accepted; moreover, the elementary steps can be realized according to different sequences (if possible) and through different procedural steps. In other terms, the problem-solving process develops along various branches. The program reacts by adapting itself to the student's choices: According to the selected solving strategy and procedural steps, it evaluates as right those answers that are consistent with the choices made and the calculations performed.

Moreover, each action by the student determines a reaction from the system that gives immediate and facilitative feedback (Shute, 2008) aimed at scaffolding the solving process of the problem, without suggesting solutions. The design of the feedback has been realized by considering the most common difficulties afflicting the learning of the content at stake (improper generalizations, calculation errors, mistakes based on incorrect definitions); moreover, it is specifically oriented to guide students to check the consistency of the given answers with the available knowledge. This promotes a meta-level learning that consists of inducing students to an *a posteriori* control and monitoring of the results obtained by the performed procedures, toward an aware view of problem-solving.

Implementation Details

The implementation of the activities and the structure of the IPSE has been carried on using some specific features of GeoGebra. First, the structure of the IPSE, envisaging the possibility to steer the student from one resource to another one, exploits the Java script supported by GeoGebra allowing to open a specific web page by clicking on a button. The learning path faced by each student is constructed by using the conditioned visualization of GeoGebra. Other GeoGebra features used for the implementation of the tasks are as follows: (i) *drag functionality*, allowing students to tidy up the sub-problems to be addressed for solving the problem and indicating by arrows corresponding elements, like elementary steps and theoretical results justifying them; (ii) *check boxes*, allowing the students to choose the solving method, by means of specific procedural steps; and (iii) *insert fields*, that students need to fill in with numeric or symbolic answers. Moreover, we used the GeoGebra graphical view to show the geometrical interpretation of the proposed problem. In this case, students are allowed and encouraged to move the graphical view to explore the geometrical meaning of the given problem (e.g. in the IPSE devoted to the extraction of a basis for a vector subspace, students can evaluate the linear dependence of specific subspaces both through their equations and the related geometrical representation).

Methods

The Case Study: Participants and Context

The explanatory case study (Cohen et al., 2007) related to our research hypothesis concerns five IPSEs devoted to specific topics in Linear Algebra. According to Cohen et al. (2007), in the following, we are going to specify the features of our explanatory case study, which are participants, context and materials. The IPSEs have been delivered to Computer Engineering freshmen, attending a face-to-face course of "Geometry, Algebra and Logic" at the University of Salerno, addressing issues of linear algebra, 2D and 3D geometry and logic. The course was taught in the second term of the academic year 2018/19, when students already had some knowledge and skills about basic calculus; it developed over 12 weeks, with three face-to-face classes (both lectures and exercise sessions) of 2 h per week. The students had also been offered resources and activities from an available online course on the Moodle platform. The resources consisted of various materials such as notes, slides and videos.

Two kinds of activities were implemented for the case study, IPSEs and Moodle workshops: (i) five IPSEs on vector spaces had been delivered as URL resources, without any time constraints, so that the students could access the resources at their own pace, and (ii) five workshop (WSs) had been submitted to the students, some of them focused on specific topics and one as a review before the mid-term exam. In the WSs, students were required to solve a problem (first phase), and they were expected to upload their productions within 4 days. It is worthwhile to note that the request of each WS consisted not only of solving the problem, but is also explicitly required for a "correct, clear and complete" solution, equipped with suitable explanations and references to the underpinning theory. Then, the platform provided each student with three solutions from peers, and a peer-review phase is envisaged (second phase), according to specific assessment criteria defined by the teacher: correctness, completeness and clearness (Albano et al., 2022).

The data collected for the case study analysis are just the students' solving processes of the problems submitted within the first phase of the WSs and devoted to the peer assessment. We remark that just the peer assessment requires students to pay much attention to communication, as they should be understood by peers and then they cannot rely on the teachers' cooperative communication.

During the course, the IPSEs and the WSs were delivered according to Figure 5. The course had two written exams, focused on solving problems: one mid-term and one final. The exams were graded along a scale from A (the best one) to F (the insufficient level).

Research Procedure and Tools

In order to validate the design of IPSE with respect to our research hypothesis, we looked at the students' productions within the WSs using the lens of Habermas' components of rational behaviour. As shown in Figure 5, the students carried out the first two WSs without having at their disposal any IPSE. We evaluated WS1 and WS2, according to the same scale used for the exam (that is from A to F). This allowed us to correlate the students' grades with their productions within the WSs, before and after the interaction with the IPSEs, and to define students' profiles. With this term, we mean the dynamic students' characteristics in their problem-solving activities in terms of strategic planning competence. In particular, we describe their evolution as improvements in different components of rational behaviour.

For the analysis, we used the data collected automatically by the platform. They consist of two different kinds: (1) log files, reporting the distribution of each student's accesses to IPSE resources, and (2) the files uploaded by students during the WSs (for our analysis, we refer to the files of the first phase, that is the solving processes of the given problems).

The former data provided information about the number of accesses done by the students to IPSE resources and allowed us to consider those students (44) who used IPSE at least two times (at least once before the first WS). This allowed us to connect the interaction with IPSE and the student's productions. The latter data allowed

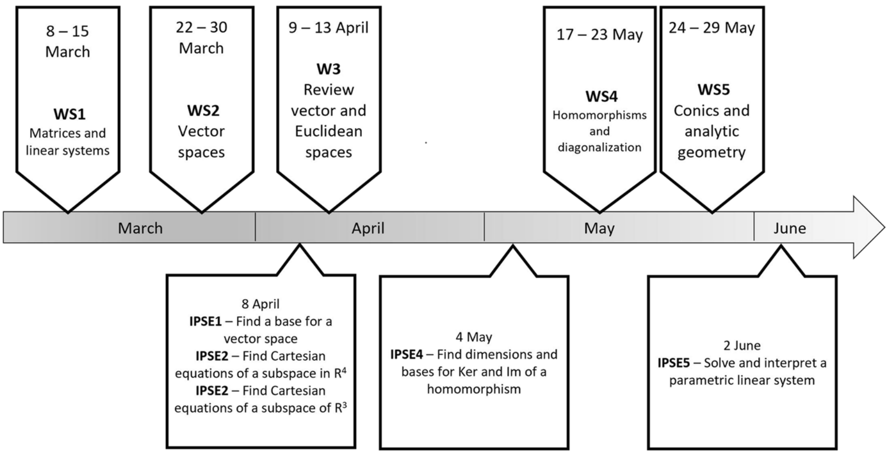


Fig. 5 The timeline of the distribution of the resource within the online course

us to analyse from the qualitative point of view (Sharma, 2013) the students' solving processes with respect to research question 2. To do that, we looked at the students' written comments on the solution they provided, that is their response to the request of being clear and of giving explanations: in other terms, we focused on their interpretation and enactment of what/why/how to solve the proposed problems and on the communication of the solving process. In this respect, we coded students' excerpts concerning the knowledge of mathematical facts as referred to the epistemic component of rationality; we coded students' excerpts concerning the actions toward a goal as referred to the teleological component of rationality; finally, we looked at the way the students communicate their problem-solving processes with reference to the communicative component of rationality.

In detail, our research procedure envisaged the repeated and separated reading and coding of the data by the researchers, until they agreed on the identification of relevant themes and the definition of students' profiles according to research question 2 and the theoretical framework. The resultant profiles, discussed in Section 7, are hence both theory- and data-driven.

Outcomes and Analysis

We analysed the productions of 44 students. The analysis allowed us to identify some students' profiles, characterized by some improvements in different components of the rational behaviour.

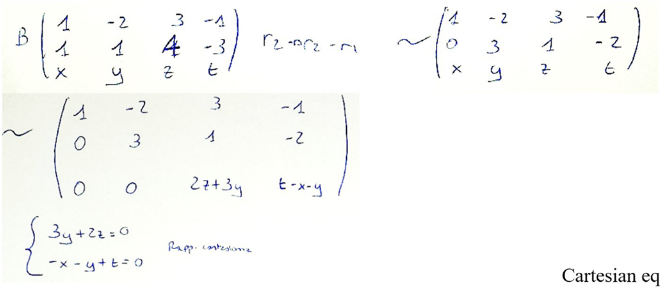
In this section, we present the protocols of Renzo, Lisa, Maria and Paolo. They have been chosen among the 44 students since they are representative of students with different grades at the mid-term exam and with productions displaying different levels of knowledge, skills and awareness before the educational experiment. The analysis of their productions allowed us to highlight a significant change in problem solving processes after the interaction with our IPSEs.

Let us see some Renzo's transcripts. In Figure 6, there is the request of the WS2, and it is shown how Renzo solves it from the operational viewpoint, by declaring how the computation will be performed (making explicit the row operations made for the row reduction). Indeed, he solves the task using correctly a given algorithm without any justification (although explicitly asked by the general request of the WS). However, he initially states what he is looking for, that is, he gives the definition of Cartesian equation for V.

Moving to WS5, handling the problem shown in Figure 7, Renzo starts to solve the task by explaining the planning of actions (*teleological component of rational behaviour*), linking to appropriate theoretical references (*epistemic component of rational behaviour*), clearly communicated (*communicative component of rational behaviour*). Note that he refers firstly to a theoretical issue, which in this particular case coincides also with an elementary step (see the first sentence of the solving process in Figure 7), performed by means of a procedural step (see the second sentence); so doing, he explains in advance his planning for the solution of the problem, and finally he proceeds towards the computations. We hypothesize that this is

Let V= < (1,-2,3,-1), (1,1,4,-3)> be a vector space of $\mathbb{R}^4$. Compute Cartesian equations for V.

For Cartesian equations for V we mean that we make use of a homogeneous linear system. We reduce [in echelon form] the matrix constituted by the vectors of V and a generic vector u=(x,y,z,t).



Cartesian equations

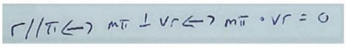
Fig. 6 The request of WS2 for Renzo and his solving process

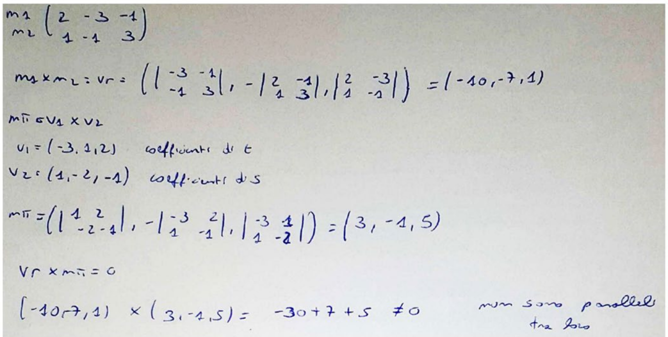
Given the line and the plane

$$\pi : \begin{cases} x = 2 - 3t + s \\ y = 1 + t - 2s \\ z = -1 + 2t - s \end{cases} , \quad r : \begin{cases} 2x - 3y - z - 1 = 0 \\ x - y + 3z - 2 = 0 \end{cases}$$

establish if r is parallel to π.

A plane and a line are parallel if and only if the direction vector of the line and the normal vector of a plane are orthogonal. That is their dot product is zero. Hence





[r and π] are not parallel

Fig. 7 The request of WS5 for Renzo and his solving process

because he considers useful the information given by the theory to solve the computational items.

Taking into account the analysis along all WSs, we notice that Renzo displays an increasing accuracy in strategic planning toward a goal and in the capability of theoretically justifying his actions. However, when he puts into practice his planning, he does not justify the elementary steps he carries out (e.g. he takes for granted that to analysing the linear independence of vectors it is possible to evaluate the rank of a suitable matrix). In other terms, the teleological component of rationality seems to be improved, while the epistemic component is fully activated for what concerns definitions and results justifying the sub-problems to be solved, but not enough for the elementary and procedural steps. This analysis suggests that the teleological component of rationality, which IPSE explicitly addresses, is predominant over the other ones; moreover, among the other components of rationality, it seems that the epistemic one prevails over the communicative one. Indeed, the student does not bother to explain his calculations to the reader. Although the change in the solving process by Renzo from WS2 to WS5 may also be partially attributed to the kind of task, we remark that the general request for all the WSs was to clearly explain and justify the solving process. In this sense, Renzo performs more effectively the proposed task in WS5, and he seems to take advantage of the structure of problem solving suggested by IPSE.

Let us now analyse the productions of Lisa. Her first submissions present overabundant theoretical references and calculations. For example, Figure 8 shows that she grounds her procedure on the definition of the basis of a vector space, and she verifies for the given set of vectors both the conditions that it is linearly independent and that it is a system of generators, without taking advantage from knowing the dimension of the vector space given by the problem. It seems that Lisa tries to maintain linguistic control over the actions she performs, exploiting the problem as an opportunity to review theoretical issues. At the same time, it seems that she does not grasp knowledge expressed by language: for example, she refers before to the notion of "linear combination", explaining it in detail as a sum of terms that are products of vectors and numbers (see Remark in item 1), and she does the same in item 2, without referring to the expression "linear combination" nor the Remark in item 1.

Her subsequent submissions are less pedantic and display increasing mathematical skills, but also increased self-confidence, control and awareness of the available tools. In the WS5, within the solving process, Lisa coordinates three different semiotic systems (algebraic, verbal, and geometric) for describing a line: she uses a system of equations (given in the problem), translates it into verbal language by interpreting geometrically each one of the equations of the system as a plane, and hence the line as an intersection of planes, and draws the planes and the line (see Figure 9).

Moreover, although Lisa's work does not show a clear plan before she performs procedural steps, she declares some progressively partial goals to be achieved. This

Let us consider the subset of $\mathbb{R}^3$: S={(-1,-2t,1), (-3t,1,-2t), (t,t,0)}.

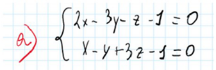
- For which values of t in $\mathbb{R}$ is S a basis for $\mathbb{R}^3$?

In order for S be a basis it is necessary:

1. Verifying the linear independence of vectors in S, i.e. the property according that no vector of the given set is obtainable as **<u>linear combination</u>** of the other vectors.
   - **<u>Remark:</u>** a linear combination is an expression formed by a sum of terms multiplied for constants.
2. Verifying that the set S is a generator for the vectors of $\mathbb{R}^3$.
   - or, in a better way, verifying that any vector v of $\mathbb{R}^3$ comes from a sum of terms which are products of scalar numbers and vectors in S.

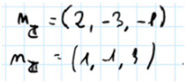
Fig. 8 The request of WS2 for Lisa and her solving process

Given the line r of Cartesian equations $\begin{cases} 2x - 3y - z - 1 = 0 \\ x - y + 3z - 1 = 0 \end{cases}$, calculate parametric equations of r.

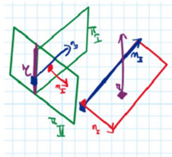


The line represented in this way is the the intersection of two planes.

- I identify the vectors n_I and n_{II}, orthogonal to the first and the second plane, respectively.



obtained by the coefficient [of] x, y, z



- The directional vector of the required line will be orthogonal to the plane formed by n_I and n_{II}.

The "vectorial product" associated with each pair of vectors is a third vector which is orthogonal to both of them.

Fig. 9 The request of WS5 for Lisa and her solving process

suggests that she sometimes follows the foreseen global sequence Plan–Justify–Perform and that sometimes she moves along a different sequence by declaring a step, justifying and performing it, then declaring the subsequent step, justifying and performing it, etc., so localizing the envisaged global sequence.

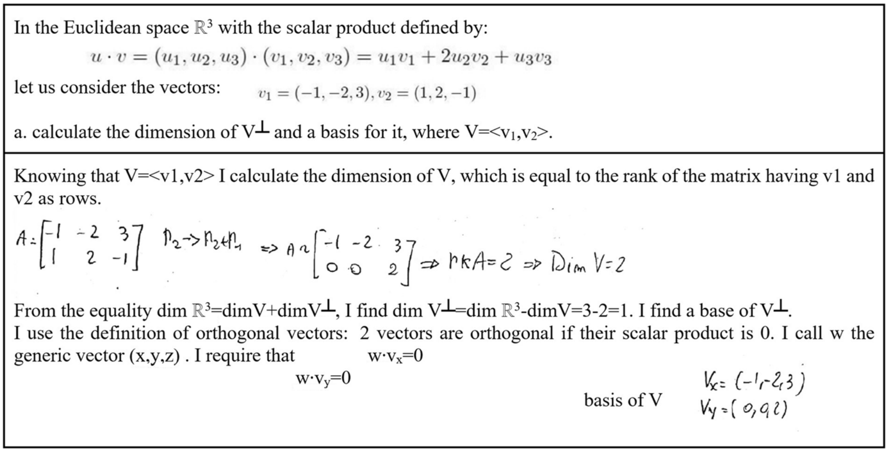
From the communicative point of view, her last deliveries assume the traits of a narration, where conceptual and procedural knowledge are harmonically integrated, the useless elements are omitted, while the crucial ones are well linked towards the final solution.

Looking at the analysis of all WSs, we may note that Lisa's transcripts are initially characterized by standardized and mnemonic executions, and they gradually show a clear improvement in terms of awareness and self-confidence, which was precisely a goal of our IPSEs. She displays a clear improvement in the epistemic component of rationality and in the control of the meanings of the mathematical contents; moreover, she progressively develops good coordination between the algebraic and geometric languages and ways of thinking. Finally, from the communicative viewpoint, Lisa's submissions become gradually more synthetic, clearer, and effective.

Let us turn now to Maria and Paolo. Within WS3, handling the exercise in Figure 10.

Maria solves the exercise, describing step by step the plan followed, clearly communicating what to do (step of the plan: "I calculate the dimension of [the vector space] V", Figure 10), how (which computations: "rank of the matrix having v1 and v2 as rows", Figure 10), together with an explanation of how to perform the declared steps (elementary row operations) and why (underpinning theory or data: "Knowing that V = <v1, v2>"). Maria seems to solve the problem according to a local sequence Plan–Justify–Perform, i.e. not explaining her plan in advance, but describing step by step what she plans, justifies and performs.

In WS4, Maria displays an analogous behaviour to that in WS3, but it is worthwhile noticing that, as a new feature, she activates explicitly an a-posteriori



In the Euclidean space $\mathbb{R}^3$ with the scalar product defined by:

$$u \cdot v = (u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + 2u_2v_2 + u_3v_3$$

let us consider the vectors: $v_1 = (-1, -2, 3), v_2 = (1, 2, -1)$

a. calculate the dimension of $V^{\perp}$ and a basis for it, where $V=<v_1,v_2>$.

Knowing that V=<v1,v2> I calculate the dimension of V, which is equal to the rank of the matrix having v1 and v2 as rows.

$A = \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ $R_2 \to R_2 + R_1$ $\Rightarrow A \sim \begin{bmatrix} -1 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow rkA = 2 \Rightarrow Dim\ V = 2$

From the equality dim $\mathbb{R}^3$=dimV+dim$V^{\perp}$, I find dim $V^{\perp}$=dim $\mathbb{R}^3$-dimV=3-2=1. I find a base of $V^{\perp}$.
I use the definition of orthogonal vectors: 2 vectors are orthogonal if their scalar product is 0. I call w the generic vector (x,y,z) . I require that $w \cdot v_x = 0$

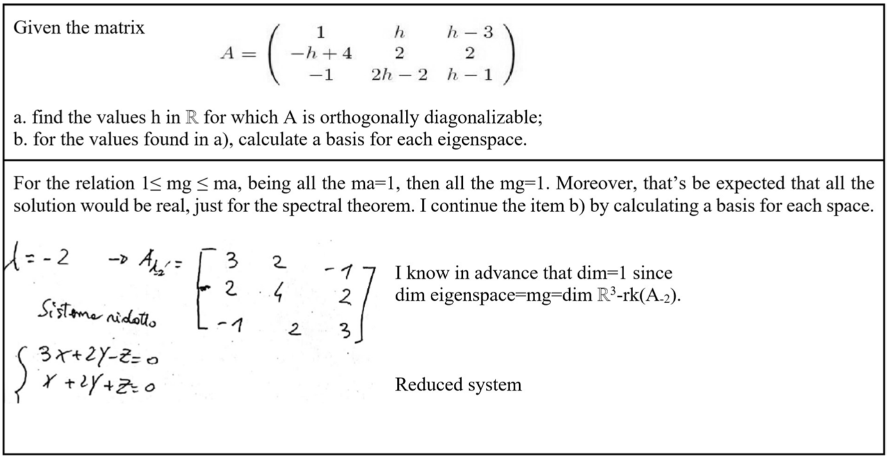
$w \cdot v_y = 0$

basis of V $V_x = (-1,-2,3)$
$V_y = (0,0,2)$

Fig. 10 The request of WS3 for Maria and her solving process

control ("that's to be expected", Figure 11), as well as anticipating some results ("I know in advance", Figure 11), by fruitfully connecting computations and theoretical results.

Let us consider now some productions by Paolo. In solving the problem of WS3 shown in Figure 11, he displays awareness and mastery of knowledge when he sets up the system in a symbolic way and then substitutes the numerical values. Moreover, he shows a strong control towards the final goal of the problem and hence a clear focus on the teleological component of rational behaviour, when he avoids solving the first equation of the system and writing the second one; he writes "I can already conclude that…" (last line in Figure 12).



Given the matrix

$$A = \begin{pmatrix} 1 & h & h-3 \\ -h+4 & 2 & 2 \\ -1 & 2h-2 & h-1 \end{pmatrix}$$

a. find the values h in $\mathbb{R}$ for which A is orthogonally diagonalizable;
b. for the values found in a), calculate a basis for each eigenspace.

For the relation 1≤ mg ≤ ma, being all the ma=1, then all the mg=1. Moreover, that's be expected that all the solution would be real, just for the spectral theorem. I continue the item b) by calculating a basis for each space.

$\lambda = -2 \to A_{\lambda_2} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 2 \\ -1 & 2 & 3 \end{bmatrix}$ I know in advance that dim=1 since dim eigenspace=mg=dim $\mathbb{R}^3$-rk(A_{-2}).

Sistema ridotto

$\begin{cases} 3x+2y-z=0 \\ x+2y+z=0 \end{cases}$ Reduced system

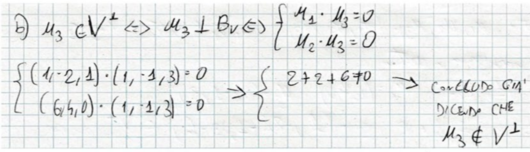
Fig. 11 The request of WS4 for Maria and her solving process

In the Euclidean space $\mathbb{R}^3$ defined by the following scalar product

$$u \cdot v = (u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = 2u_1v_1 + u_2v_2 + 2u_3v_3$$

let us consider the vectors $u_1 = (1, -2, 1)$, $u_2 = (6, 4, 0)$, $u_3 = (1, -1, 3)$

b) Establish if u_3 belongs to $V^{\perp}$.



I can already conclude that $u_3 \notin V^{\perp}$.

Fig. 12 The request of WS3 for Paolo and his solving process

Concerning the problem handled in WS5 (Figure 13), Paolo's excerpts (Figure 12 and Figure 13) allow us to highlight how an anticipatory thought (Boero, 2002) arises, through which Paolo predicts the possible results of a computation he is going to do. Indeed, before identifying the reciprocal position of two planes, he lists the different cases that can occur (see lines "If rk(A) …" in Figure 13). Moreover, he easily switches between the geometric and algebraic representations (see line 1 in Paolo's solving process in Figure 13, together with the conclusions derived from "if rk(A) …"), firstly reducing the problem of the mutual positions of geometrical loci in the Euclidean space to the calculation of ranks of matrices, and then associating to each possible result of this calculation the corresponding geometric situation.

Given the planes

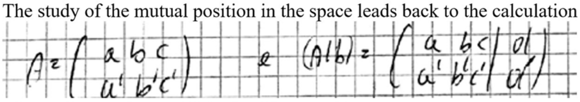
$$\pi : \begin{cases} x = -3 - t + s \\ y = 2 - 2t - s \\ z = 1 - t + 4s \end{cases}, \qquad \pi' : \begin{cases} x = 2 - 3t - s \\ y = -4 + 2t - s \\ z = -3 + t + 2s \end{cases}$$

and the line

$$r : \begin{cases} x = -2 + t \\ y = 1 - 4t \\ z = 3 - t \end{cases},$$

determine if π is parallel to π' and, if not, find the line s $s = \pi \cap \pi'$.

The study of the mutual position in the space leads back to the calculation of the rank of the matrices



where a,b,c are the coefficients of the equations.

We need to study the compatibility of the system $\begin{cases} \pi = 0 \\ \pi' = 0 \end{cases}$.

If rk(A)=2, then rk(A|b)=2

- the system is possible and has ∞^1 solutions → the planes are incident and the solutions are the points of the line generated by the intersection of the planes.

If rk(A)=1

- if rk(A|b) is 2, the system is impossible → planes parallel
- if rk(A|b)=1 → ∞^2 solutions → planes coincident

Fig. 13 The request of WS5 for Paolo and his solving process

Maria and Paolo display analogous behaviour, specifically for what concerns the progressive development of a metacognitive and strategic control on the elementary and procedural steps performed. From the first deliveries to the last ones, Maria and Paolo develop an anticipatory thought, allowing them to predict possible results, and manifest the attitude to check a-posteriori the calculations done by comparing them with the available information. Moreover, they display a progressive transition from the global sequence Plan–Justify–Perform to a repeated local sequence Plan–Justify–Perform for each procedural step (Figure 15).

Discussion

The analysis of the students' productions showed the positive effects of their interactions with IPSE and their evolution in problem-solving capability in terms of strategic planning and awareness of what/why/how to do. According to the IPSE structure, the students seem to have acquired the methodological attitude promoted by IPSE: The more they interact with IPSE, the more they generally propose the planning of the steps needed to solve the problem, justify and carry out them by means of suitable procedures. In this respect, the replicability of IPSE, allowing us to design five different IPSEs, is an essential aspect. The analysis of the students' productions and the reference to the theory informed us for identifying some students' profiles, depending on the students' progress in the use of the Habermas' components of rationality and their capability to face a task through strategic planning. Hence, these profiles are both theory- and data-driven.

A first profile is constituted by *low achievers improving mainly on epistemic and communicative sides*. These students achieve F- or E-grade in the first WSs; Lisa is one of the 14 students with this profile. They generally solve the first assignments in a standardized and procedural way and display many difficulties concerning the meanings of the mathematical concepts at stake. Typically, they make many calculations, probably due to theoretical lacks or inability to connect the knowledge. After the interaction with our IPSEs, the students progressively build self-confidence and awareness about the implications of the calculations performed, improve their control on the meanings of the mathematical concepts and develop the capability to integrate the algebraic and the geometrical level of the tasks. Moreover, some of these students develop an anticipatory thought displayed by the attempt to predict the possible results, before implementing the calculation procedures; they also activate an *a posteriori* control of the consistency of the obtained results. On the basis of our analysis, these students generally reach a medium grade in the final exams, showing a remarkable improvement mainly for what concerns the use of epistemic and communicative components of rationality.

A second profile collects *average achieving students mainly improving in strategic planning*. These students achieve D- or C-grade in the first WSs. They display a remarkable improvement in their attitude to planning strategically the resolution of a task and to going into the plan by performing each procedural step. Renzo can be considered a representative of the 24 students with this profile. The way for solving the tasks applied by these students passes from a mainly procedural level, although quite justified by theoretical notions and results, to a more conceptual one, in which the algorithms carried out are motivated and oriented to a specific and declared goal, which is related to the request of the task. The learning path of the students with this profile displays a progressively arising predominance of the teleological component of rational behaviour, to which our IPSE is specifically addressed, with respect to the epistemic and the communicative ones.

Finally, the third profile we have highlighted is represented by *high achievers displaying a dynamic intertwinement of the three components of rational behaviour, metacognitive control and anticipatory thought*. These students achieve B- or A-grade in the first WSs; Maria and Paolo represent two of five students in this category. These students improve their metacognitive and strategic control of the procedures performed and their consequences toward a specific goal. They progressively activate forms of anticipatory thought and *a posteriori* control on the results obtained by the applied procedures and clearly communicate their solving process, so effectively intertwining all the three components of rational behaviour. On one hand, the anticipatory thought allows students to exploit the available knowledge (given by the theory or acquired by the calculations) to achieve the goal in a smart way; on the other hand, the metacognitive control allows students to check the appropriateness of the obtained solution with respect to the context and to the request. The diagram in Figure 14 shows the back and forth movement between the available knowledge and the set goal so that the former is continuously interpreted according to the latter and vice versa. This movement is made concrete by the use of anticipatory thought and metacognitive control and their relationship with the interrelated components of rational behaviour.

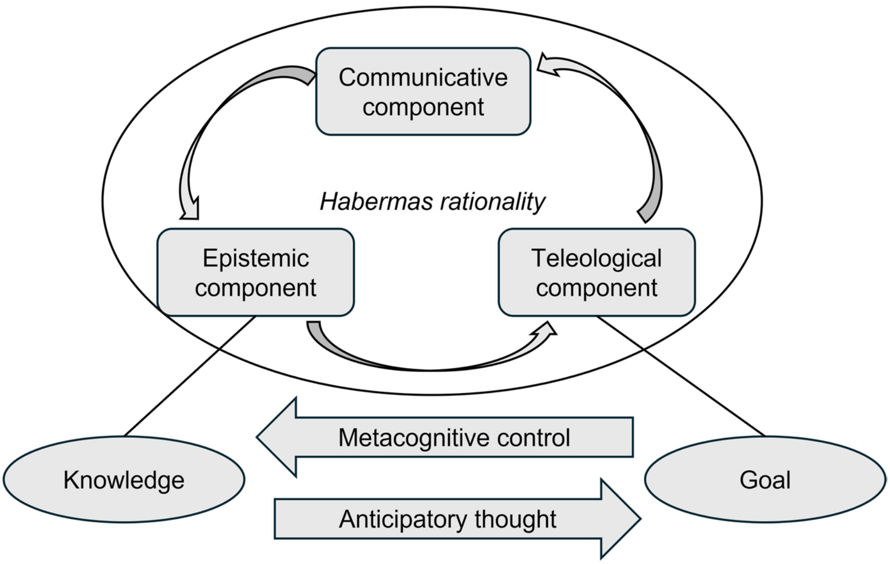
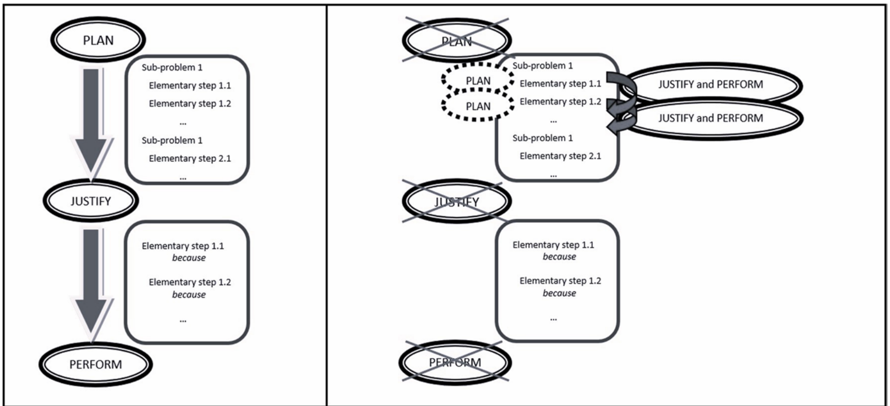


Fig. 14 The diagram showing the intertwinement among Habermas' rational components, anticipatory thought and metacognitive control

A further characteristic of this profile, already emerged in some cases of the previous profile, is the change of working style from the enactment of the global sequence Plan–Justify–Perform induced by IPSE (Figure 15a) to a local variation of it (Figure 15b) or mixed schemes. This is displayed in task resolutions providing firstly a general planning and then going through each procedural step of it; after the interaction with our IPSEs, the solving style of these students presents mainly the description of one local goal and the performance of the procedural steps, one by one, theoretically justified, for reaching it.

We are aware that the described changes in the students' productions could depend on different causes, such as the kind of the task, their commitment and progress in understanding the topics. However, taking into account that all the WSs required students for a justified and clearly communicated solving process for the proposed task, the presented productions reproduce the structure of IPSE, and, in this respect, they seem to suggest the impact of the interaction with IPSE on students' improvements.

Although all students interact without any difficulty with IPSE, understanding the tasks to be accomplished, and the positive outcomes of our case study, some criticalities arose. Some recurring difficulties emerged from the students' protocols. They mainly concern the meaning of specific mathematical concepts, like the Cartesian and parametric equations of a vector subspace; moreover, some students lack mastery on the variation of parameters. These aspects, linked with misconceptions discussed in the literature, suggest a redesign of our IPSE or the design of specific IPSEs aimed to minimize the emerged difficulties. In particular, we would like to deepen the potentiality of our IPSE in order to support the students' handling of the generality of a parameter and apply the format of IPSE to learning and producing proofs, in order to foster the students' conceptual understanding of proofs as complex objects, for which is needed the integration of all the Habermas' component of rational behaviour.



a) global sequence | b) local sequence

Fig. 15 Schemes of possible working styles associated to IPSE

Conclusion

The study outcomes seem to suggest that the students with the third profile demonstrate the very dynamic nature of the Habermas' rationality: for these students, the three components of rational knowing, acting and communicating are well integrated and dialectically interact in the practice of rationality itself and its development. Indeed, these students show a dynamic integration between what is available from the performed calculations and the final aim; they display a continuous search for a meta-control on the evidence following from the implemented procedures and for strategic use of them with respect to a specific goal. These students' behaviour puts the epistemic and the teleological components together, moving continuously back and forth from the available knowledge (both that given from the task and that arising from the performed calculations) and the set goal, and interpreting the former in light of the latter, also paying attention to the communicative choices.

The conducted case study suggested some answers to our research questions and opened many others.

For the future, it would be interesting to investigate some further issues emerged by the outcomes of this study:

- Educational possible effects of the identification of students' profile: how can the student and the teacher take advantage of knowing the student's profile?
- Educational strategies to improve the students' profile: how can the teacher favour the student's moving to a more advanced profile?
- Strategies for deepening the profiling of the students: on one hand, how can we use technological tools to make the students' profiling automatic? On the other hand, how can we make the students' profiling more fitting by integrating the collected data with students' interviews?

Finally, more theoretical and experimental research is needed in order to understand the contexts favouring the full realization of the dynamicity of the Habermas' rationality in mathematical activity and its connection with heuristics in problem-solving. Educational experiments involving numerous samples and carried out with heterogeneous activities in different areas of Mathematics should shed light on ways in which the dynamic nature of rationality displays, its dependence on the students' profile, and to what extent it relies on the students' interaction with IPSEs or on changes of the didactical contract.

Author Contribution The authors contributed equally to the study conception, design, data analysis, to the development of the manuscript and both of them read and approved the final version.

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Data Availability Datasets generated during the current study are not publicly available because they contain identifiable information. Transcripts created during analysis are available from the corresponding author on reasonable request.

Declarations

Competing interest The authors declare no competing interests.

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