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The κ-logistic growth model. Qualitative and quantitative dynamics

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ABSTRACT

The κ-exponential function, representing a generalization of the exponential function, has been firstly introduced in physics, and, then, it has been considered in a noteworthy number of fields because of its ability to take rare events into account. Among the possible applications of this function, one of particular interest is in economics in which rare events may consist in natural disasters, such as earthquakes that reduce the supply of capital, or epidemics or other external shocks influencing the supply of intermediate inputs, human or physical capital. Starting from the κ-exponential function, the κ-logistic function, which is a generalization of the sigmoidal function, can be obtained and used to describe production functions in a unique setting to take into account (1) several shapes usually considered in economics (i.e. concave and non-concave production functions), (2) economies at different development levels, and, (3) the possible occurrence of rare events. In this paper, we investigate the economic growth model as proposed by Böhm and Kaas (2000), wherein the production function utilizes the κ-logistic function. We provide theoretical results confirmed by extensive computational experiments and in line with economic literature showing that a poverty trap may emerge together with fluctuations, multistability and complex dynamics.

1. Introduction

Economic growth is a fundamental branch of Macroeconomics. Its importance is confirmed by the plethora of available models. A significant number of these models trace their origins to two pioneering works: the optimal-growth model by Ramsey [20] and the Solow–Swan model (see e.g. [25,26]). Later works have explored more complex settings. Among all, the paper published by Böhm and Kaas [4] is a fundamental milestone. The authors examine the evolution of capital per-capita in a discrete-time setup by assuming differential savings between workers and shareholders and considering a production function satisfying the so-called weak Inada conditions. These conditions ensure the existence of a unique (positive) steady state.

After Böhm and Kaas [4] introduced their work, numerous models have been proposed to consider different assumptions, as for instance the weak Inada conditions are not always satisfied or the elasticity of substitution between production factors (measuring the ease by which one factor can be substituted for another, for a given level of output) is not constant.

Grassetti et al. [11] were the first to propose the use of a shifted Cobb–Douglas to describe the production function, i.e. a technology that states the existence of a minimum level of capital needed before making returns. Such a function is continuous, non-strictly concave, and non-differentiable; the weak Inada conditions do not hold and the elasticity of substitution between production factors depends on the capital per-capita level. Their model is able to exhibit a poverty trap, as it often occurs in less developed economies.

Another production function is the Constant Elasticity of Substitution (CES) production function as firstly introduced by Solow [25]. Brianzoni et al. [5] revised the Böhm and Kaas [4] growth model by considering a CES production function that does not

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satisfy the weak Inada conditions even though the elasticity of substitution between production factors remains constant. In such model fluctuations may emerge and, in addition, the authors show that a crucial role is played by the substitutability between production factors, i.e. if the elasticity of substitution between production factors is greater than one, the economy may converge to the steady state.

Brianzoni et al. [7] also considered a variant of the model moving to a Variable Elasticity of Substitution (VES) production function as firstly proposed by Revankar [22]. In such a case the substitutability between production factors is not constant and multistability may emerge, i.e. coexistence of attractors, and the economy may fluctuate over time when the elasticity of substitution between production factors is not too high.

In order to take into account the case in which a minimum level of capital per-capita may be reached before decreasing returns are observed (as occurs in less-developed economies), Brianzoni et al. [6] proposed a sigmoidal production function. This function is positive, strictly increasing, and always convex–concave, it does not satisfy the weak Inada conditions and the elasticity of substitution between production factors is variable. The maximum production level the economy can achieve, i.e. the upper-bound of the production function, varies depending on some parameters also influencing the elasticity of substitution between production factors. Such a sigmoidal production function has been previously proposed by Capasso et al. [8] in a similar formulation but with continuous time. Similar to what occurs with the shifted Cobb–Douglas a poverty trap may emerge, i.e. policies to push economies at a minimum development level are necessary to enter in a region characterized by economic growth.

Finally, the well-known Leontief technology has been considered by Böhm and Kaas [4] and Tramontana and Avrutin [27]. The obtained growth model results non-continuous, fluctuations may emerge and the structure of related bifurcations is described.

The novelty of the present work is the use of the κ-logistic function, denoted by σ_κ (see [17]), to describe the production function in the Böhm and Kaas [4] growth model. Such a function is a generalization of the well-known sigmoidal logistic function, frequently used in machine learning (see e.g. [3]) and it is obtained from the κ-exponential function (or, more concisely, $\exp_\kappa$) introduced by Kaniadakis [15,16].

The $\exp_\kappa$ function presents the property that when the parameter κ approaches zero, it tends to the $\exp$ function; otherwise, with κ large enough, it tends to plus or minus infinity like a power function. Its popularity is due to the capability to take rare events into account. In fact, while ordinary events follow an exponential law, rare events are characterized by a Pareto (or power-tail) law. Although the $\exp_\kappa$ originated in physics, its applications embrace a noteworthy number of fields, such as economy (see e.g. [9]), finance (see e.g. [19]), epidemiology (see e.g. [18]) and computer science (see e.g. [23]). More in detail, rare events affecting economic growth may consist in natural disasters, such as earthquakes that reduce the supply of capital, or epidemics or other external shocks influencing the supply of intermediate inputs, human or physical capital. As Benson and Clay [2] argue, rare events have the potential to affect economic growth through their effect on output, investment, fiscal balances and the balance of payments. More recent contributions concerning rare events in economics are in [1,13].

Starting from the $\exp_\kappa$ function, the κ-logistic function has been proposed by Kaniadakis [17] in the field of statistical physics and, as previously mentioned, it generalizes the logistic sigmoidal function, i.e. it maps the set of the real numbers onto the set $(0, 1)$ and shows as an S-shape like the sigmoidal function proposed by Capasso et al. [8]. However, unlike the sigmoidal production function considered by Brianzoni et al. [6] the κ-logistic function offers a number of advantages. First, as previously discussed, it is able to take rare events into account because of the presence of the $\exp_\kappa$ function in its definition. Secondly, it is able to describe production functions that are both concave or convex–concave (like the sigmoidal one). Finally, the formulation proposed in this paper is able to consider different levels of upper bounds for production by varying an opportune parameter thus taking into account economies at different development levels. Differently from the previously used sigmoidal production function, we consider an upper bound that can be fixed independently of the elasticity of substitution level between production factors.

By combining analytical tools and numerical experiments we show that the discrete time growth model can behave in different ways, depending on parameter values. According to these values, the map describing the difference equation can be either concave, non-concave and increasing, or bimodal (so that fluctuations may emerge).

We then provide some theoretical conditions and numerical experiments to determine the number of equilibrium points and their stability. We show that multistability may emerge, thus making the choice of the initial condition crucial. In particular, with S-shape production functions, economies at low development level or experiencing sufficiently high probability of incurring in rare events may converge to the poverty trap. However, if the economy is sufficiently developed, then concave production functions produce economic growth patterns converging to the unique positive steady state even in the presence of relevant rare events. Finally, long-run dynamics can also produce economic cycles or complex behavior: fluctuations are more likely to emerge when the elasticity of substitution between production factors is not too high, in line with the results coming from economic literature.

The paper is organized as follows. In Section 2 we briefly recall the different production functions proposed in the above-mentioned contributions and summarize their properties. In Section 3, we revisit the $\exp_\kappa$ introduced by Kaniadakis [15,16], obtain the σ_κ sigmoidal function and describe its properties. In Section 4, we introduce the κ-logistic growth model and present conditions for the existence and local stability of equilibria, while Section 5 is dedicated to computational experiments, which also corroborate the theoretical insights. Section 6 concludes our paper.

2. Previous production function formulations

In this section, we aim at providing a more detailed account of the production function formulations recalled in the introduction. We denote with x_t, $t \in \mathbb{N}$, the capital per-capita level at time t.

Following Böhm and Kaas [4], the map describing the evolution of capital per-capita with differential savings between workers and shareholders is described by the following difference equation:

$$x_{t+1} = \frac{1}{1+n}\left[(1-\delta)x_t + s_w w(x_t) + s_r \pi(x_t)\right], \tag{1}$$

where $n \geq 0$ is the labor force growth rate, $\delta \in [0, 1]$ is the capital depreciation rate, $s_w \in [0, 1]$ is the saving rate for workers and $s_r \in [0, 1]$ is the saving rate for shareholders.

We denote by $\mathbb{R}^+$ the set of non-negative real numbers and by $\mathbb{R}^{++}$ the set of positive real numbers and consider a production function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Then the wage rate $w(x_t)$ equals the marginal product of labor, i.e. $w(x_t) = f(x_t) - x_t f'(x_t)$, while the profit share is given by $\pi(x_t) = x_t f'(x_t)$ and represents the marginal product of capital received by stakeholders.

Another assumption made by Böhm and Kaas [4] is that the production function satisfies the weak Inada conditions, i.e. $f \in C^2\left(\mathbb{R}^+\right)$, f is strictly monotonically increasing, strictly concave and such that $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$, and $\lim_{x\to 0} \frac{f(x)}{x} = +\infty$.

As it has been underlined, after the Böhm and Kaas [4] model, several technologies have been proposed to describe the relationship between capital per-capita and production, not necessarily verifying the weak Inada conditions, and with constant or variable elasticity of substitution between production factors. We recall some examples.

The CES production function used by Brianzoni et al. [5] and defined as

$$f(x_t) = \left(1 + x_t^{\rho}\right)^{\frac{1}{\rho}},$$

with $\rho < 1$ and $\rho \neq 0$, is characterized by constant elasticity of substitution.

Differently, the shifted Cobb–Douglas function defined as

$$f(x_t) = \begin{cases} 0 & 0 \leq x_t \leq x_c \\ A\left(x_t - x_c\right)^{\alpha} & x_t > x_c, \end{cases}$$

with $A > 0$, $\alpha \in (0, 1)$, and $x_c \geq 0$; or the VES production function introduced by Revankar [22] and given by

$$f(x_t) = A x_t^{a\gamma}\left(1 + b a x_t\right)^{(1-a)\gamma},$$

with $A > 0$, $a \in (0, 1)$, $b \geq -1 \wedge b \neq 0$, $1/x_t \geq -b$, and $\gamma = 1$; or, finally, the sigmoidal production function utilized by Brianzoni et al. [6] and defined as

$$f(x_t) = \frac{\alpha x_t^p}{1 + \beta x_t^p},$$

with $\alpha > 0$, $\beta > 0$, and $p \geq 2$; are examples of production functions with non-constant elasticity of substitution.

Finally, the Leontief technology function considered by Tramontana and Avrutin [27] is

$$f(x_t) = \min\{a x_t, b\} + c,$$

where a, b, and c are positive constants is an example of a discontinuous production function. The main results concerning the use of the above mentioned formulations have been recalled in the Introduction.

3. The modified κ-logistic production function

The $\exp_\kappa$ function is a function mapping $\mathbb{R}$ onto $\mathbb{R}^{++}$ given by

$$\exp_\kappa(x) = \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{\frac{1}{\kappa}}, \tag{2}$$

where $\kappa \in \mathbb{R} - \{0\}$ is a fixed parameter.

This function was firstly introduced by Kaniadakis [15,16] to propose a new entropy in physics. Its primary feature is to generalize the well-known exponential function $y = e^x$. Indeed, it can be shown that $\lim_{\kappa\to 0} \exp_\kappa(x) = e^x$.

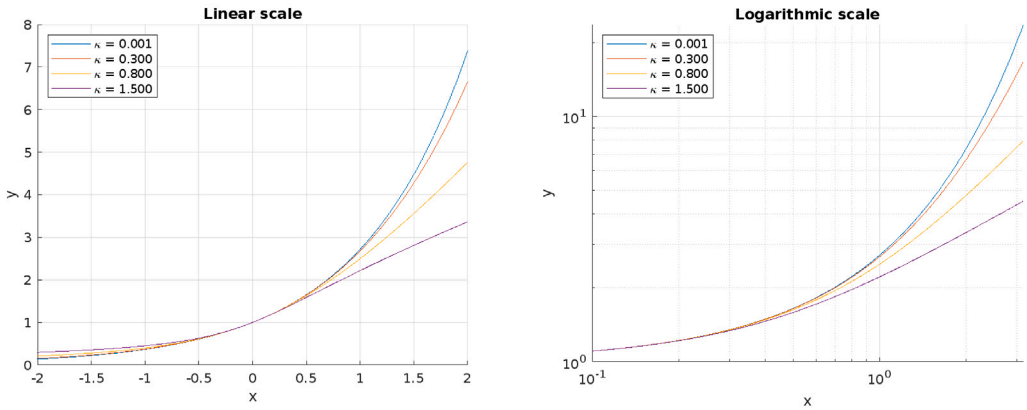
Apart from this property, the $\exp_\kappa$ function shares several properties with the exponential function, as detailed next. (1) From (2), it follows that $\exp_\kappa(x) > 0\ \forall x \in \mathbb{R}$. (2) By substituting x with 0 in (2), it results $\exp_\kappa(0) = 1$. (3) Rationalizing the term $\sqrt{1 + \kappa^2 x^2} + \kappa x$ in (2), then $\exp_\kappa(x)\exp_\kappa(-x) = 1$. (4) Similarly, it can be shown that $\exp_{-\kappa}(x) = \exp_\kappa(x)$. Thus, we can focus solely on positive values of κ.

The first derivative of the $\exp_\kappa$ function is:

$$\frac{d}{dx}\exp_\kappa(x) = \frac{\exp_\kappa(x)}{\sqrt{1 + \kappa^2 x^2}}, \tag{3}$$

which is always positive. Therefore, like the ordinary exponential function, the $\exp_\kappa$ function is strictly monotonic increasing. Additionally, it is straightforward to prove that $\lim_{x\to-\infty} \exp_\kappa(x) = 0$ and $\lim_{x\to+\infty} \exp_\kappa(x) = +\infty$. Fig. 1 shows the $\exp_\kappa$ function, both in linear and logarithmic scale, for different values of κ.

From the aforementioned properties and from Fig. 1, it is clear that the $\exp_\kappa$ shares many similarities with the exponential function. However, unlike the ordinary exponential function, the $\exp_\kappa$ function can be concave for large values of the independent variable when $\kappa > 1$. Another fundamental feature of the $\exp_\kappa$ function is the capability to handle rare events. Indeed, while ordinary



(a) (b)

Fig. 1. The $\exp_\kappa$ function for different values of κ, both in linear (panel a) and logarithmic scale (panel b).

events follow an exponential law, rare events obey a power-tail (or Pareto) distribution as $x \to \pm\infty$. In support of this affirmation, another property of the $\exp_\kappa$ function is that

$$\exp_\kappa(x) \sim |2\kappa x|^{\pm\frac{1}{\kappa}} \quad \text{when} \quad x \to \pm\infty. \tag{4}$$

This property, combined with the fact that $\lim_{\kappa\to 0} \exp_\kappa(x) = e^x$ shows how the $\exp_\kappa$ function is able to manage rare events. This is expressed by (4) when $|\kappa| > \epsilon$, with ϵ large enough. Conversely, if κ is small enough, the $\exp_\kappa$ function behaves like the standard exponential function, modeling the case without rare events. This adaptability explains the widespread use of the $\exp_\kappa$. From the exponential function, the well-known logistic function (see e.g. [3,14,21]), also termed sigmoidal due to its S-shape, can be obtained. It is defined as follows: $\sigma : \mathbb{R} \to (0, 1)$, and

$$\sigma(x) = \frac{1}{1 + e^{-x}}. \tag{5}$$

The popularity of this function lies in the following fundamental properties. The first one is its ability to map the set $\mathbb{R}$ to the interval $(0, 1)$, thus transforming a number into a probability. For this reason, this function is widely used in the field of machine learning. Furthermore, as we show next, since its first derivative is always positive, this mapping is increasing, meaning that larger numbers correspond to probabilities closer to one and vice versa. Finally, its characteristic S-shaped curve makes it suitable not only for probabilistic contexts but also for applications that, as we underline in this article, go beyond the realm of machine learning. This function has properties such as $\lim_{x\to-\infty} \sigma(x) = 0$ and $\lim_{x\to+\infty} \sigma(x) = 1$. Moreover, it can be proven [21] that:

$$\frac{d}{dx}\sigma(x) = \sigma(x)(1 - \sigma(x)). \tag{6}$$

Since the image set of σ is $(0, 1)$, from (6) we deduce that $\sigma'(x) > 0$. Therefore, the logistic function maps all real numbers to values between 0 and 1 in an increasing monotonic manner.

When the κ-exponential function is taken into account, we move to the κ-logistic function given by:

$$\sigma_\kappa(x) = \frac{1}{1 + \exp_\kappa(-x)}. \tag{7}$$

It is worth noting that, being the σ_κ function derived from the $\exp_\kappa$ function, the domain of the parameter κ is $\mathbb{R} - \{0\}$ as well. Although σ_κ is not defined for $\kappa = 0$, with a slight abuse of notation (but for the sake of simplicity) we can extend its definition for $\kappa = 0$ exploiting the aforementioned fact that

$$\lim_{\kappa\to 0} \exp_\kappa(x) = e^x.$$

In this way, it is straightforward to verify that

$$\lim_{\kappa\to 0} \sigma_\kappa(x) = \sigma(x).$$

Thus, we can extend with continuity the definition of the σ_κ function in order to also include the case when $\kappa = 0$ and with the convention that $\sigma_0(x) = \sigma(x)$. Even more so, a similar extension can also be made for the $\exp_\kappa$ function with the obvious convention that $\exp_0(x) = e^x$. In Proposition 1, we outline several properties of the κ-logistic function.

Proposition 1. *The σ_κ function satisfies the following properties:*

(a) $\lim_{x\to-\infty} \sigma_\kappa(x) = 0$.

(b) $\lim_{x\to+\infty}\sigma_\kappa(x) = 1$.
(c) $\sigma_\kappa(x) \in (0,1)\ \ \forall\, x \in \mathbb{R}$.
(d) $\sigma_{-\kappa}(x) = \sigma_\kappa(x)\ \ \forall\, \kappa \in \mathbb{R}$.
(e) $\frac{d}{dx}\sigma_\kappa(x) = \frac{1}{\sqrt{1+\kappa^2x^2}}\sigma_\kappa(x)\left(1-\sigma_\kappa(x)\right)$.
(f) *σ_κ is a strictly monotonic increasing function.*
(g) *$x_f = 0$ is an inflection point for σ_κ. Specifically, σ_κ is convex for $x < x_f$ and concave for $x > x_f$.*
(h) *A line intersects the σ_κ function in at most three points.*
(i) *σ'_κ is an even function and its maximum is at the origin with a value of $\frac{1}{4}$.*

Proof. Points (a) and (b) are easily demonstrated by leveraging the property that $\lim_{x\to-\infty}\exp_\kappa(-x) = +\infty$ and $\lim_{x\to+\infty}\exp_\kappa(-x) = 0$.

We prove point (c) in two steps: first we show that $\sigma_\kappa(x) > 0\ \forall\, x \in \mathbb{R}$ and then we demonstrate that $\sigma_\kappa(x)$ is always less than one. Since $\exp_\kappa(-x) > 0$, the denominator of (7) is always positive, as its reciprocal. Moreover, it follows that $1 + \exp_\kappa(-x) > 1$, implying $\frac{1}{1+\exp_\kappa(-x)} < 1$.

Point (d) can be easily proven by leveraging the aforementioned property that the $\exp_\kappa$ function is even with respect to κ. We have:

$$\sigma_{-\kappa}(x) = \frac{1}{1+\exp_{-\kappa}(-x)} = \frac{1}{1+\exp_\kappa(-x)} = \sigma_\kappa(x).$$

Point (e) can be proven as follows:

$$\begin{aligned}\frac{d}{dx}\sigma_\kappa(x) &= -\frac{1}{\left[1+\left(\sqrt{1+\kappa^2x^2}-\kappa x\right)^{\frac{1}{\kappa}}\right]^2}\frac{d}{dx}\left(\sqrt{1+\kappa^2x^2}-\kappa x\right)^{\frac{1}{\kappa}}\\ &= \sigma_\kappa(x)\,\sigma_\kappa(x)\,\frac{\exp_\kappa(-x)}{\sqrt{1+\kappa^2x^2}} = \frac{1}{\sqrt{1+\kappa^2x^2}}\,\sigma_\kappa(x)\,\frac{1+\exp_\kappa(-x)-1}{1+\exp_\kappa(-x)}\\ &= \frac{1}{\sqrt{1+\kappa^2x^2}}\,\left(1-\sigma_\kappa(x)\right)\,\sigma_\kappa(x).\end{aligned}$$

Point (f) directly follows from points (c) and (e). We prove point (g) by computing σ''_κ. We have:

$$\sigma''_\kappa(x) = \frac{\sqrt{1+\kappa^2x^2}\,\frac{d}{dx}\left(\sigma_k(x)\left(1-\sigma_\kappa(x)\right)\right) - \sigma_k(x)\left(1-\sigma_\kappa(x)\right)\frac{\kappa^2x}{\sqrt{1+\kappa^2x^2}}}{1+\kappa^2x^2} \tag{8}$$

Using the product rule, it can be convenient to express $\frac{d}{dx}\left(\sigma_k(x)\left(1-\sigma_\kappa(x)\right)\right)$ as $\sigma'_\kappa(x)\left(1-2\sigma_k(x)\right)$. Similarly, using the result from point (e), we can write $\sigma_k(x)\left(1-\sigma_\kappa(x)\right)\frac{\kappa^2x}{\sqrt{1+\kappa^2x^2}}$ as $\kappa^2x\sigma'_\kappa(x)$. With these premises, we can reformulate (8) as:

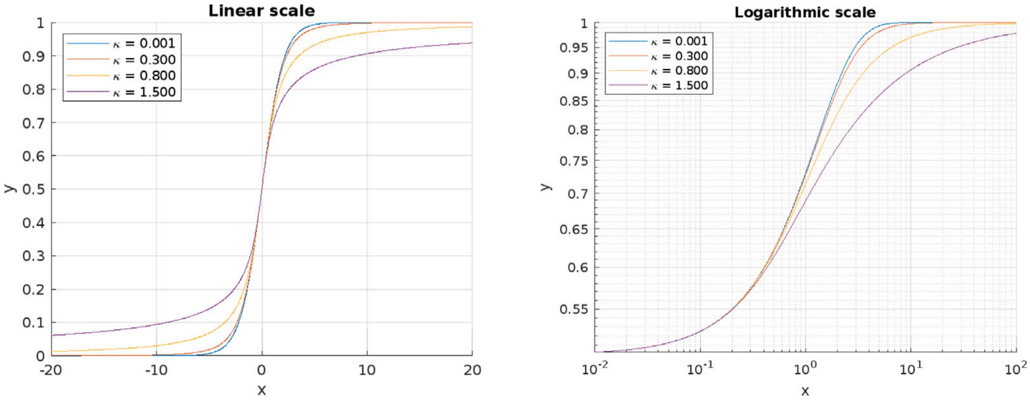
$$\sigma''_\kappa(x) = \left[\frac{1-2\sigma_\kappa(x)}{\sqrt{1+\kappa^2x^2}} - \frac{\kappa^2x}{1+\kappa^2x^2}\right]\sigma'_\kappa(x). \tag{9}$$

Evaluating (9) at $x = 0$ and considering that $\sigma_\kappa(0) = \frac{1}{2}$, it follows that $\sigma''_\kappa(0) = 0$. By point (f), $\sigma'_\kappa(x) > 0\ \forall x \in \mathbb{R}$. Thus, to study the convexity or the concavity of σ_κ, it suffices to examine the sign of:

$$\frac{1-2\sigma_\kappa(x)}{\sqrt{1+\kappa^2x^2}} - \frac{\kappa^2x}{1+\kappa^2x^2}. \tag{10}$$

Putting together the facts that (1) σ_κ is bounded between 0 and 1, (2) $\sigma_\kappa(0) = \frac{1}{2}$, and (3) σ_κ is a monotonic strictly increasing function, we deduce that $0 < \sigma_\kappa(x) < \frac{1}{2}\,\forall\, x < 0$. Specifically, since $\sigma_\kappa(x) < \frac{1}{2}\,\forall\, x < 0$, then $1 - 2\sigma_\kappa(x) > 0\,\forall\, x < 0$. Consequently, the expression in (10) is positive for all $x < 0$ implying σ_κ is convex in $\mathbb{R}^-$. The proof that σ_κ is concave in $\mathbb{R}^+$ proceeds similarly, noting that $\sigma_\kappa(x) > \frac{1}{2}\,\forall\, x > 0$.

To prove point (h), we leverage the obvious property that a line meets a convex or concave function that is neither a line nor a linear piece-wise function in at most two points. By point (g), σ_κ is convex on $\mathbb{R}^-$ and concave on $\mathbb{R}^+$. Thus, if a line intersects in the second quadrant the σ_κ function at zero or one point, then it will intersect the same function in the first quadrant at most twice. Likewise, if a line intersects in the first quadrant the σ_κ function in zero or one point, then it will intersect the same function in the second quadrant in at most two points. The last thing to show consists in proving that if a line meets in the second quadrant the σ_κ function at two points, then it will intersect the same function in the first quadrant at one more point and vice versa. To prove this, let $\left(x_1, \sigma_\kappa(x_1)\right)$ and $\left(x_2, \sigma_\kappa(x_2)\right)$ with $x_1 < x_2 \le 0$ be these points. In this region, σ_κ lies below the line in the interval (x_1, x_2) and above the line in the intervals $(-\infty, x_1)$ and $(x_2, 0]$. Since σ_κ intersects the y axis above the line (or on the line if its equation is $y = mx + \frac{1}{2}$) and σ_κ is concave on $\mathbb{R}^+$ and $\lim_{x\to+\infty}\sigma'_\kappa(x) = 0$, the function will intersect the line at one more point in the first quadrant. In a similar way, it can be proved that if the line intersects σ_κ at two points in the first quadrant, it will intersect σ_κ at one more point in the convex region.



(a) (b)

Fig. 2. The σ_κ function for different values of κ, both in linear (panel a) and logarithmic scale (panel b).

Finally, to prove (i), let $x \in \mathbb{R}$. Then:

$$\begin{aligned}\sigma_\kappa(-x) &= \frac{1}{1+\exp_\kappa(x)} = \frac{1}{1+\frac{1}{\exp_\kappa(-x)}} = \frac{\exp_\kappa(-x)}{1+\exp_\kappa(-x)}\\ &= \frac{1+\exp_\kappa(-x)-1}{1+\exp_\kappa(-x)} = 1-\sigma_\kappa(x).\end{aligned} \tag{11}$$

This property is useful to show that:

$$\sigma'_\kappa(x) = \frac{1}{\sqrt{1+\kappa^2x^2}}\sigma_\kappa(x)\left(1-\sigma_\kappa(x)\right) = \frac{1}{\sqrt{1+\kappa^2x^2}}\sigma_\kappa(x)\sigma_\kappa(-x) = \sigma'_\kappa(-x).$$

From the considerations in point (g), we know that σ''_κ is zero only at $x=0$. Since $\sigma''_\kappa(x) > 0\ \forall\, x<0$ and $\sigma''_\kappa(x) < 0\ \forall\, x>0$, it follows that $x=0$ is a global maximum for σ'_κ with value $\sigma'_\kappa(0) = \frac{1}{4}$. □

It is worth noting that, as already expressed in the proof of point (e) of Proposition 1, the first derivative of the κ-logistic function is always positive. Thus, similar to the logistic function, the κ-logistic function maps all real numbers onto $(0,1)$ in a strictly monotonic increasing manner. In Fig. 2, we depict the graph of the κ-logistic function for various values of κ, both in linear and logarithmic scale.

The goal of this paper is to propose a modified version of the κ-logistic function as a potential expression for the production function, mirroring the shape of the sigmoidal function introduced by Capasso et al. [8] in the continuous time model and used by Brianzoni et al. [6] in the discrete-time setup, but with additional desired properties. More in detail we aim at considering the following new points.

- Suggest a generalized production function able to describe both strictly concave technologies (as for instance the Cobb–Douglas standard production function) or a convex–concave production function (as for instance the sigmoidal production function previously introduced in the literature that cannot be always concave).
- Take into account rare events, as for instance, in economics, natural disasters, such as earthquakes that reduce the supply of capital, or epidemics or other external shocks influencing the supply of intermediate inputs, human or physical capital.
- Consider economies at different development levels according to an upper bound to the maximum production level that can be reached by the economy and that is not linked to the elasticity of substitution between production factors.

For this purpose, we construct the following modified κ-logistic production function, namely $f(x_t)$, such that $f(x_t)$ passes through the origin, it is strictly increasing but bounded from above by a positive constant M and it can be both convex and convex–concave depending on the sign of a constant x_c.

Then high values of M correspond to more developed economies, positive values of x_c are associated to non-strictly concave production functions and, finally, higher κ values are associated to situations in which rare events are more likely to emerge. The proposed production function, called modified κ-logistic, takes into account all these points, and it is given by:

$$f(x_t) = M\frac{\sigma_\kappa(x_t-x_c)-\sigma_\kappa(-x_c)}{1-\sigma_\kappa(-x_c)}. \tag{12}$$

Given that the production function is derived from the σ_κ function, the domain of the parameter κ remains $\mathbb{R}-\{0\}$. However, as done with the $\exp_\kappa$ and σ_κ, we can continuously extend the domain of κ to include the case $\kappa=0$ because

$$\lim_{\kappa\to 0} f(x_t) = M\frac{\sigma(x_t-x_c)-\sigma(-x_c)}{1-\sigma(-x_c)}.$$

In Proposition 2, we outline two properties of the proposed modified κ-logistic function.

Proposition 2. *The modified κ-logistic function has the following characteristics:*

(a) f *is convex on* $(-\infty, x_c)$ *and concave on* $(x_c, +\infty)$.
(b) $\forall\, m, q \in \mathbb{R}$, *the equation* $f(x_t) = mx_t + q$ *has at most three roots.*

Proof. Point (a) directly stems from the fact that

$$f''(x_t) = \frac{M}{1-\sigma_k(-x_c)}\sigma_k''(x_t - x_c).$$

Point (b) can be addressed by manipulating the equation $f(x_t) = mx_t + q$. After some rearrangements, one derives

$$\sigma_k(x_t - x_c) = m\frac{1-\sigma_k(-x_c)}{M}x_t + q\frac{1-\sigma_k(-x_c)}{M} + \sigma_k(-x_c). \tag{13}$$

Substituting $x := x_t - x_c$ and rearranging in (13), we encounter the scenario of intersecting the σ_k function with a line. By point (h) in Proposition 1, this equation has at most three solutions. □

In light of the result proved in point (a) of Proposition 2, it is important to make a clarification. The natural domain of f is $\mathbb{R}$ and within this domain, the statement in point (a) always holds. However, in our application, x_t represents the capital per capita, which cannot be negative in economic applications. Thus, henceforth in our model, we consider f restricted to the domain $\mathbb{R}^+$ and the following corollary trivially holds.

Corollary 1. *Let* f *be defined as in (12) over the domain* $\mathbb{R}^+$. *If* $x_c > 0$ *then* f *is convex in* $[0, x_c)$ *and concave in* $(x_c, +\infty)$. *Vice versa, if* $x_c \le 0$, *then* f *is concave in* $\mathbb{R}^+$.

Proof. It is a straightforward consequence of Proposition 2, point (a). □

Regarding the elasticity of substitution of the modified κ-logistic production function denoted by E_f, we recall that it measures the ease by which one factor can be substituted for another, for a given level of output. As long as the substitutability between production function increases, then E_f increases: for instance in the case of a linear technology where capital and labor are perfectly substitutable, the elasticity of substitution tends to $+\infty$, while, when the two production factors are perfect complements, the technology has fixed-proportions and $E_f = 0$. However, while some technologies are characterized by constant elasticity of substitution (as for instance the Cobb–Douglas having $E_f = 1$ or the CES function with fixed degree of substitutability between capital and labor), in other cases economies are better described by technologies with variable elasticity of substitution between production factors, i.e. substitutability between capital and labor moves depending on changes in the economy's per capita capital level. In this last case, the elasticity of substitution varies depending on the ratio between capital and labor (as for instance for the VES production function).

The elasticity of substitution for a continuous and twice differentiable function as defined by Sato and Hoffman [24] is given by

$$E_f(x_t) = -\frac{f'(x_t)\left[f(x_t) - x_t f'(x_t)\right]}{x_t f(x_t) f''(x_t)},$$

then, the modified κ-logistic function belongs to the set of variable elasticity of substitution production function, as we obtain

$$E_f(x_t) = -\frac{\frac{1}{x_t} + \frac{\sigma_\kappa(x_t-x_c)}{\sqrt{1+\kappa^2(x_t-x_c)^2}}\frac{\sigma_\kappa(x_t-x_c)-1}{\sigma_\kappa(x_t-x_c)-\sigma_\kappa(-x_c)}}{\frac{1-2\sigma_\kappa(x_t-x_c)}{\sqrt{1+\kappa^2(x_t-x_c)^2}} - \frac{\kappa^2(x_t-x_c)}{1+\kappa^2(x_t-x_c)^2}}.$$

Notice that both parameters κ and x_c influence the elasticity of substitution between production factors. Anyway, the relationship between them and E_f as x_t varies is quite complex to be analytically investigated. Some interesting findings can be observed using numerical simulations. For instance, various computational tests have shown that, by fixing $x_c \ge 0$, the elasticity function exhibits a monotonically increasing trend w.r.t. κ as long as $x > x_c$ and a monotonically decreasing trend when $x < x_c$.

4. The κ-logistic growth model

Starting from (1), we assume that the wage rate $w(x_t)$ is equal to the marginal product of labor, i.e., $w(x_t) = f(x_t) - x_t f'(x_t)$, while the profit share is equal to the marginal product of capital i.e., $\pi(x_t) := x_t f'(x_t)$. Under this assumption, (1) becomes $x_{t+1} = F(x_t)$, where $F : \mathbb{R}^+ \to \mathbb{R}$ is defined as follows:

$$F(x_t) = \frac{1}{1+n}\left[(1-\delta)x_t + s_w f(x_t) + x_t f'(x_t)(s_r - s_w)\right]. \tag{14}$$

Taking into account point (e) of Proposition 1, (14) becomes:

$$F(x_t) = \frac{1}{1+n}\left\{(1-\delta)x_t + s_w M\frac{\sigma_\kappa(x_t-x_c) - \sigma_\kappa(-x_c)}{1-\sigma_\kappa(-x_c)}\right.$$

$$+\frac{(s_r - s_w)M}{1-\sigma_\kappa(-x_c)} \frac{\sigma_\kappa(x_t - x_c)\left(1-\sigma_\kappa(x_t - x_c)\right)x_t}{\sqrt{1+\kappa^2(x_t-x_c)^2}}\Bigg\}. \tag{15}$$

Our analysis begins with the equilibrium points of the map $x_{t+1} = F(x_t)$, i.e., those points x^* satisfying the equation $x^* = F(x^*)$. Due to the analytical complexity of (15), it is not possible to determine its equilibrium points exactly, for at least two reasons: (i) the number of parameters involved is extensive, and (ii) we are dealing with a transcendental equation. However, in Proposition 3, we provide conditions that ensure the existence or non-existence of equilibrium points.

Proposition 3. *Consider the map $x_{t+1} = F(x_t)$, with F defined as in (15). Then,*

(a) $x^* = 0$ is always an equilibrium point to the map.
(b) If $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, there exists at least a $x^* > 0$ such that $F(x^*) = x^*$.
(c) If $s_r = s_w$, there are at most two positive equilibrium points. Moreover, under this condition:

(c.1) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists only one positive equilibrium point.
(c.2) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) \le (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, there are no positive equilibrium points.

(d) There exists $\epsilon > 0$ such that if $|s_r - s_w| < \epsilon$, then there are at most two positive equilibrium points. Moreover, under this condition:

(d.1) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists only one positive equilibrium point.
(d.2) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) \le (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there are no positive equilibrium points.

(e) If $s_w = 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} \le \frac{1}{4}$, then there exists at most two positive equilibrium points. In particular, if $s_w = 0$, $x_c > 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} = \frac{1}{4}$, there exists exactly one positive equilibrium point equal to $x^* = x_c$. Vice versa, if $s_w = 0$, $x_c \le 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} = \frac{1}{4}$, there are no positive equilibrium points. Moreover, if $s_w = 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} > \frac{1}{4}$, then there are no positive equilibrium points.
(f) There exists $\underline{s}_w > 0$ such that for all $s_w \in (0, \underline{s}_w)$, the map has at most two positive equilibrium points.

Proof. The proof of (a) is straightforward: since $f(0) = 0$, it follows that $F(0) = 0$.

(b) We look for an oblique asymptote of F in the form $x_{t+1} = mx_t + q$. From Calculus, we have:

$$m = \lim_{x_t\to+\infty} \frac{F(x_t)}{x_t} = \frac{1-\delta}{1+n}, \tag{16}$$

and

$$q = \lim_{x_t\to+\infty} \left(F(x_t) - mx_t\right) = \frac{s_w M}{1+n}. \tag{17}$$

From (14), we have:

$$F'(x_t) = \frac{1}{1+n}\left[1-\delta + s_r f'(x_t) + (s_r - s_w)x_t f''(x_t)\right]. \tag{18}$$

In particular,

$$F'(0) = \frac{1}{1+n}\left[1-\delta + s_r f'(0)\right] = \frac{1}{1+n}\left[1-\delta + s_r M\frac{\sigma_\kappa(-x_c)}{\sqrt{1+\kappa^2 x_c^2}}\right].$$

Since the slope m in (16) is always less than one, then there exists a neighborhood $\overline{N}$ of $+\infty$ such that F lies below the bisector of the first quadrant for each $x_t \in \overline{N}$. In particular, there exists a $\overline{x} \in \overline{N}$ such that $F\left(\overline{x}\right) < \overline{x}$. If parameters are chosen such that $F'(0) > 1$, i.e., such that $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists a right neighborhood of the origin $\underline{N}$ such that $F(x_t) > x_t\, \forall\, x_t \in \underline{N}$. In particular, there exists a $\underline{x} \in \underline{N}$ such that $F(\underline{x}) > \underline{x}$. Consider the function $G(x_t) = F(x_t) - x_t$ on the interval $\left[\underline{x}, \overline{x}\right]$. Since G is continuous, $G(\underline{x}) > 0$ and $G(\overline{x}) < 0$, then, by Bolzano's theorem, there exists at least a $x^* \in \left[\underline{x}, \overline{x}\right]$ such that $G(x^*) = 0$. In other words, there exists at least a positive equilibrium point for the map.

(c) If $s_r = s_w$, the map becomes

$$x_{t+1} = \frac{1}{1+n}\left[(1-\delta)x_t + s_w f(x_t)\right]. \tag{19}$$

Thus, the equilibrium points are the roots of the equation

$$f(x^*) = \frac{n+\delta}{s_w}x^*. \tag{20}$$

By Proposition 2, (20) has at most three roots. Since, as already proved, one of them is $x^* = 0$, it follows that there are at most two positive equilibrium points. In this particular case, deriving F in (19) respectively once and twice, we get:

$$F'(x_t) = \frac{1}{1+n}\left[1-\delta+s_w f'(x_t)\right] = \frac{1}{1+n}\left[1-\delta+s_r\frac{M}{1-\sigma_\kappa(-x_c)}\sigma'_\kappa(x_t-x_c)\right], \tag{21}$$

and

$$F''(x_t) = \frac{s_r}{1+n}\frac{M}{1-\sigma_\kappa(-x_c)}\sigma''_\kappa(x_t-x_c). \tag{22}$$

If $x_c < 0$, then, by Proposition 1, σ_κ is concave for all $x_t \geq 0$ and, by (22), so is F. Moreover, from (21) we see that F is a strictly increasing monotonic function because F' is always positive. Thus, following a similar approach as previously done in (b), if $F'(0)$ computed in (21) is greater than 1, there exists a positive equilibrium point. Otherwise, there are no positive equilibrium points.

(d) It is a generalization of (c). We have:

$$x_t f'(x_t) = \frac{M}{1-\sigma_\kappa(-x_c)}\frac{x_t}{\sqrt{1+\kappa^2(x_t-x_c)^2}}\sigma(x_t-x_c)\left(1-\sigma_\kappa(x_t-x_c)\right). \tag{23}$$

The function defined in (23) and restricted to the domain $\mathbb{R}^+$ has no vertical asymptotes and $\lim_{x_t\to+\infty}\left(x_t f'(x_t)\right) = 0$, therefore it is limited. Let $K > 0$ be an upper bound to this function and let $\gamma > 0$. If $|s_r - s_w| < \frac{\gamma}{K}$, then

$$\underline{F}_\gamma(x_t) < F(x_t) < \overline{F}_\gamma(x_t) \tag{24}$$

with

$$\underline{F}_\gamma(x_t) = \frac{1}{1+n}\left[(1-\delta)x_t + s_w f(x_t) - \gamma\right]$$

and

$$\overline{F}_\gamma(x_t) = \frac{1}{1+n}\left[(1-\delta)x_t + s_w f(x_t) + \gamma\right].$$

Both equations $\underline{F}_\gamma(x^*) = x^*$ and $\overline{F}_\gamma(x^*) = x^*$ reduce to intersecting $f(x_t)$ with a line and, as proved in Proposition 2, the related equation has at most three solutions. We have that

$$\lim_{\gamma\to0^+}\underline{F}_\gamma(x_t) = \lim_{\gamma\to0^+}\overline{F}_\gamma(x_t) = \frac{1}{1+n}\left[(1-\delta)x_t + s_w f(x_t)\right]. \tag{25}$$

As already proved in point (c), the equation

$$x^* = \frac{1}{1+n}\left[(1-\delta)x^* + s_w f\left(x^*\right)\right]$$

has at most three solutions. Therefore, putting these considerations all together, there is a small-enough γ such that if $|s_r - s_w| < \frac{\gamma}{K}$, then F has at most three equilibrium points. It is enough to choose $\epsilon = \frac{\gamma}{K}$ and the thesis holds. Points (d.1) and (d.2) are carried out as points (c.1) and (c.2) by observing that

$$F'(x_t) = \frac{1}{1+n}\left[1-\delta+s_r f'(x_t) + (s_r - s_w)x_t f''(x_t)\right], \tag{26}$$

and

$$F''(x_t) = \frac{1}{1+n}\left[(2s_r - s_w)f''(x_t) + (s_r - s_w)x_t f'''(x_t)\right]. \tag{27}$$

If ϵ is small enough, then it is possible to make F' always positive. If $x_c < 0$ then, by Proposition 2, f is concave and so f'' is negative. We have:

$$2s_r - s_w = s_r + s_r - s_w = s_r \pm \epsilon$$

So, if ϵ is small enough, the term $(2s_r - s_w)f''(x_t)$ is negative. Moreover, $x_t f'''(x_t)$ is limited $\forall x_t \geq 0$. To see this, after some algebra, it is possible to prove that

$$\sigma'''_\kappa(x) = \left\{-\frac{2\sigma_\kappa(x)\left(1-\sigma_\kappa(x)\right)}{1+\kappa^2x^2} - \frac{\left(1-2\sigma_\kappa(x)\right)\kappa^2 x}{\left(1+\kappa^2x^2\right)^{\frac{3}{2}}} - \kappa^2\frac{1-\kappa^2x^2}{\left(1+\kappa^2x^2\right)^2} + \left[\frac{1-2\sigma_\kappa(x)}{\sqrt{1+\kappa^2x^2}} - \frac{\kappa^2 x}{1+\kappa^2x^2}\right]^2\right\}\sigma'_\kappa(x). \tag{28}$$

It is easy to see that the function $x\sigma'''_\kappa(x)$ is bounded in $\mathbb{R}^+$ because there are no vertical asymptotes and $\lim_{x\to+\infty}\left(x\sigma'''_\kappa(x)\right) = 0$. Therefore, the function $x_t f'''(x_t)$ is bounded as well. Thus, not only is it possible to choose an ϵ such that the term $(2s_r - s_w)f''(x_t)$ is negative but also such that, in magnitude, it is bigger than $|s_r - s_w|x_t f'''(x_t) = \epsilon x_t f'''(x_t)$. Consequently, under these circumstances, if $x_c < 0$, F is concave and analogous considerations to points (c.1) and (c.2) hold.

(e) If $s_w = 0$, the map becomes:

$$x_{t+1} = \frac{1}{1+n}\left[(1-\delta)x_t + s_r x_t\frac{M}{1-\sigma_\kappa(-x_c)}\sigma'_\kappa(x_t-x_c)\right]. \tag{29}$$

In addition to the equilibrium point $x^* = 0$, the other equilibrium points of (29) can be found solving the equation

$$\sigma'_\kappa(x_t - x_c) = \frac{(n+\delta)\left(1 - \sigma_\kappa(-x_c)\right)}{M s_r}. \tag{30}$$

By Proposition 1, the maximum value of σ'_κ is $\frac{1}{4}$ and it is reached when the argument of σ'_κ is equal to zero. Therefore, if $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{M s_r} > \frac{1}{4}$, there are no positive equilibrium points. If $x_c > 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{M s_r} = \frac{1}{4}$, there is exactly one positive equilibrium point for $x^* = x_c$. But if $x_c \leq 0$ there are no positive equilibrium points. If $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{M s_r} < \frac{1}{4}$, (30) has two distinct roots that are positive depending on the value of x_c. Therefore, in this case, there are at most two positive equilibrium points.

(f) After some algebra, the map $x^* = F(x^*)$ can be reformulated as

$$\sigma'_\kappa\left(x^* - x_c\right) = \frac{(n+\delta)\left(1 - \sigma_\kappa(-x_c)\right)}{M(s_r - s_w)} - \frac{s_w}{s_r - s_w}\frac{\sigma_\kappa\left(x^* - x_c\right) - \sigma_\kappa(-x_c)}{x^*}. \tag{31}$$

Let

$$F_\epsilon(x_t) = \frac{(n+\delta)\left(1 - \sigma_\kappa(-x_c)\right)}{M(s_r - \epsilon)} - \frac{\epsilon}{s_r - \epsilon}\frac{\sigma_\kappa\left(x_t - x_c\right) - \sigma_\kappa(-x_c)}{x_t}.$$

This function is bounded because

$$\lim_{x_t \to 0^+} \frac{\sigma_\kappa\left(x_t - x_c\right) - \sigma_\kappa(-x_c)}{x_t} = \sigma'_\kappa(-x_c)$$

and, as proven in Proposition 1, $0 < \sigma'_\kappa(x) < \frac{1}{4}\ \forall\, x \in \mathbb{R}$. Therefore,

$$\lim_{\epsilon \to 0^+} F_\epsilon(x_t) = \frac{(n+\delta)\left(1 - \sigma_\kappa(-x_c)\right)}{M s_r}.$$

But this coincides with the case treated in point (e) which has at most two positive equilibrium points. Consequently, we can find a small-enough $\underline{s}_w$ such that, for all $s_w \in (0, \underline{s}_w)$, the map has at most two positive equilibrium points. □

According to Proposition 3, the growth model may admit multiple equilibria so that the investigation of their stability properties become crucial. Unfortunately, it is not possible to qualify the nature of the equilibrium points due to the number of parameters involved and it is only possible to identify the origin in a precise way.

Hence in Proposition 4, we present a result concerning the asymptotic stability of the trivial equilibrium point.

The proof of Proposition 4 relies on two theorems that can be found in [10], pages 27–30, and we restate adopting them to our notation. The first theorem (on page 27) states that if F is continuously differentiable at an equilibrium point x^* and if $|F'(x^*)| < 1$, then x^* is asymptotically stable. Alternatively, under the same assumption about F, if $|F'(x^*)| > 1$, then x^* is unstable. Although this theorem is fundamentally important, it does not encompass the case when $|F'(x^*)| = 1$. This case is partially covered by the second theorem (on page 29) that states that if $F'(x^*) = 1$ (and F is differentiable up to the third order), then: (a) if $F''(x^*) \neq 0$, then x^* is unstable, (b) if $F''(x^*) = 0$ and $F'''(x^*) > 0$ then x^* is unstable. Finally, (c) if $F''(x^*) = 0$ and $F'''(x^*) < 0$ then x^* is asymptotically stable.

Proposition 4. *Consider the map $x_{t+1} = F(x_t)$, with F defined as in (15) with the equilibrium point at the origin.*

(a) If $s_r M \sigma_\kappa(-x_c) < (n+\delta)\sqrt{1 + \kappa^2 x_c^2}$, then the origin is asymptotically stable.
(b) If $s_r M \sigma_\kappa(-x_c) > (n+\delta)\sqrt{1 + \kappa^2 x_c^2}$, then the origin is unstable.
(c) If $s_r M \sigma_\kappa(-x_c) = (n+\delta)\sqrt{1 + \kappa^2 x_c^2}$ and $x_c \neq 0$ and $s_w \neq 2 s_r$, then the origin is unstable.
(d) If $s_r M \sigma_\kappa(-x_c) = (n+\delta)\sqrt{1 + \kappa^2 x_c^2}$, $x_c = 0$, and $3 s_r - 2 s_w > 0$ then the origin is asymptotically stable.
(e) If $s_r M \sigma_\kappa(-x_c) = (n+\delta)\sqrt{1 + \kappa^2 x_c^2}$, $x_c = 0$, and $3 s_r - 2 s_w < 0$ then the origin is unstable.

Proof. (a) Substituting $x_t = 0$ into (18), we have:

$$F'(0) = \frac{1}{1+n}\left[1 - \delta + s_r f'(0)\right] \tag{32}$$

From Proposition 1 and the definition of f, we have:

$$f'(x_t) = \frac{M}{1 - \sigma_\kappa(-x_c)}\sigma'_\kappa(x_t - x_c) = \frac{M}{1 - \sigma_\kappa(-x_c)}\frac{\sigma_\kappa(x_t - x_c)\left(1 - \sigma_\kappa(x_t - x_c)\right)}{\sqrt{1 + \kappa^2\left(x_t - x_c\right)^2}}. \tag{33}$$

Thus,

$$f'(0) = \frac{M\sigma_\kappa(-x_c)}{\sqrt{1 + \kappa^2 x_c^2}}. \tag{34}$$

The thesis holds by imposing $F'(0) < 1$.

(b) This point is proved similarly to point (a), this time imposing $F'(0) > 1$.

(c) If $s_r M\sigma_\kappa(-x_c) = (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, we are in the case where $F'(0) = 1$. In this limit case, we need to evaluate $F''(0)$. From (27), we have:

$$F''(0) = \frac{1}{1+n}\left[(2s_r - s_w)f''(0)\right]. \tag{35}$$

Considering the expression of f and using (9), we find:

$$F''(0) = \frac{1}{1+n}(2s_r - s_w)\frac{M\sigma_\kappa(-x_c)}{\sqrt{1+\kappa^2 x_c^2}}\left[\frac{1-2\sigma_\kappa(-x_c)}{\sqrt{1+\kappa^2 x_c^2}} + \frac{\kappa^2 x_c}{1+\kappa^2 x_c^2}\right] \tag{36}$$

The conclusion follows by noting that the term in the square brackets equals zero only when $x_c = 0$. Thus, if $2s_r - s_w$ and $x_c \neq 0$, then $F''(0) \neq 0$ and the origin is unstable.

To prove points (d) and (e), we differentiate (27) to get:

$$F'''(x_t) = \frac{1}{1+n}\left[(3s_r - 2s_w)f'''(x_t) + (s_r - s_w)x_t f^{IV}(x_t)\right]. \tag{37}$$

From (37), we deduce:

$$F'''(0) = \frac{1}{1+n}(3s_r - 2s_w)f'''(0). \tag{38}$$

Recalling (28), if $x_c = 0$ then (38) becomes:

$$F'''(0) = -\frac{1}{2}\left(\frac{1}{2}+\kappa^2\right)\frac{M}{1+n}(3s_r - 2s_w), \tag{39}$$

and this completes the proof because the sign of $F'''(0)$ is determined by $(3s_r - 2s_w)$ □

Although Proposition 4 states the nature of the equilibrium point at the origin in various circumstances, unfortunately, it does not contemplate all possible cases. For example, nothing is said when $x_c = 0$ and $3s_r - 2s_w = 0$. These particular cases require the use of the Schwarzian derivative that has to be evaluated also in (28) for the general case. These investigations are far from trivial. In essence, it is challenging to ascertain the sign of (28) and, even more so, of the Schwarzian derivative.

In any case, Proposition 4 point (a) is very interesting from an economic point of view, stating a result in line with previous literature (see e.g. [6,12]). More precisely, some economies may incur in a poverty trap, where there is a set of initial conditions leading to trajectories converging to the origin. This situation arises in economies characterized by a low development level (i.e., when M is sufficiently small), a sigmoidal technology requiring a high minimum capital per-capita level before entering the decreasing return region (i.e., with x_c sufficiently high), or when rare events are likely to occur (i.e., when κ is high enough).

As far as the existence of non-trivial steady state is concerned, by combining Proposition 3 with Proposition 4, the following Corollary 2 holds.

Corollary 2. *If $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists at least a positive equilibrium point and the origin is unstable.*

Proof. It is enough to consider point (b) in Proposition 3 and point (b) in Proposition 4. □

According to Corollary 2, developed economies with concave production technology and with low probability to incur in rare events cannot fail in the poverty trap.

We proceed with our study about equilibrium points in Proposition 5. As it is usual in literature, we assume $s_r \geq s_w$, i.e. shareholders save more than workers. To preserve the economic meaning of the system, this new condition prevents the unrealistic case where the map yields negative values for some $x_t \in \mathbb{R}^+$. To see this, it is enough to consider the term $f'(x_t)$ in (14). This term is never negative since

$$f'(x_t) = \frac{M}{1-\sigma_\kappa(-x_c)}\sigma'_\kappa(x_t - x_c) \geq 0.$$

The following Proposition 5 shows that, under particular circumstances, x_c and the development level of the economy measured by M play a crucial role in the existence of non-trivial equilibrium points. Moreover, it also studies the nature of the non-trivial equilibrium point for a concave production function confirming previous results in economic literature, that is concave production functions (i.e. $x_c < 0$) produce economic growth patterns converging to the unique positive steady state.

Proposition 5. *Consider the map $x_{t+1} = F(x_t)$, with F defined as in (15) and with $s_r \geq s_w$.*

(a) There exist $\overline{x}_c$ and $\overline{M} = M(\overline{x}_c) > 0$ such that for all $x_c > \overline{x}_c$ and $0 < M < \overline{M}$ the origin is the sole equilibrium point and it is globally stable.

(b) The positive equilibrium point arising in cases (c.1) and (d.1) in Proposition 3 attracts all trajectories starting from positive initial conditions.

Table 1
Parameter values for the analyzed scenarios.

Scenario	M	x_c	κ	s_r	s_w	δ	n
1	1	−2	0.5	1	0.2	0.2	0.5
2	10	−2	0.5	1	0.2	0.2	0.5
3	1	2	0.5	1	0.2	0.2	0.5
4	25	4	0.8	0.2	0.2	0.2	0.5
5	25	4	0.8	0.25	0.2	0.2	0.5
6	2.5	5	0.8	1	0.2	0.2	0.5
7	10	2	0.5	1	0.2	0.2	0.5
8	3.00941946	4.76959597	0.70003226	1	0.2	0.2	0.5
9	6	6	0.8	1	0.2	0.2	0.5
10	10	10	0.2	0.7	0.1	0.2	0.5

Proof. (a) In a neighborhood of $+\infty$, say $(\tilde{x}, +\infty)$, the map lies below the bisector of the first quadrant, regardless of the parameter values. Thus, for the remainder of the proof, we focus on the behavior of the map in the interval $[0, \tilde{x})$. According to (15), the map is composed of a linear term (with a slope less than one) plus a second term proportional to $M\frac{\sigma_\kappa(x_t-x_c)-\sigma_\kappa(-x_c)}{1-\sigma_\kappa(-x_c)}$ and a third term proportional to $\frac{M}{1-\sigma_\kappa(-x_c)}\frac{\sigma_\kappa(x_t-x_c)(1-\sigma_\kappa(x_t-x_c))x_t}{\sqrt{1+\kappa^2(x_t-x_c)^2}}$. If we demonstrate that both the second and third terms can be made sufficiently small, the thesis holds. The denominator $1-\sigma_\kappa(-x_c)$ in the second and third terms cannot be arbitrarily close to zero because, by hypothesis, $x_c > \overline{x}_c$. Let $\epsilon \in (0,1)$ and $\overline{M} = \frac{\epsilon}{2}\left(1-\sigma_\kappa(-\overline{x}_c)\right)$. Applying the triangle inequality to the second term and considering that 1 is an upper bound to the σ_κ function, we have:

$$\left|M\frac{\sigma_\kappa(x_t-x_c)-\sigma_\kappa(-x_c)}{1-\sigma_\kappa(-x_c)}\right| \le M\frac{\sigma_\kappa(x_t-x_c)+\sigma_\kappa(-x_c)}{1-\sigma_\kappa(-x_c)} < \frac{2\overline{M}}{1-\sigma_\kappa(-x_c)} = \epsilon.$$

The third term is also bounded because there are no vertical asymptotes and we are not in a neighborhood of $+\infty$. Therefore, following the same logic as with the second term, we can also make the third term arbitrarily close to zero. As a result, for all $x_t > 0$, $F(x_t) < x_t$. Furthermore, the incremental ratio of F at the origin is $\frac{F(h)}{h}$ and is between zero and one. Thus, $0 < F'(0) < 1$ and so the origin is globally asymptotically stable.

Proof of (b). Let x_1 be the positive equilibrium point. Then, $F(x_1) = x_1$. We are dealing with a concave monotonous increasing function. Let $h > 0$ be small enough. In a right neighborhood of x_1, we have $F(x_1+h) < x_1+h$. In this neighborhood, the incremental ratio at x_1 is:

$$\frac{F(x_1+h)-F(x_1)}{h} = \frac{F(x_1+h)-x_1}{h} < 1.$$

Moreover, since $F(x_1+h) > F(x_1)$, it follows that the incremental ratio is also positive. A similar argument holds for a left neighborhood of x_1. Thus, $0 < F'(x_1) < 1$ and x_1 is locally stable. However, since by Proposition 4 the origin is unstable and there are no other equilibrium points, it follows that x_1 attracts all trajectories starting from positive initial conditions. □

As outlined in Proposition 3, the model can exhibit multiple equilibrium points. As proved in Proposition 4, the origin can be locally stable. As a result, the necessary conditions for the coexistence of attractors are met. Multistability, i.e., situations where more than one attractor is present, must be further explored. Such a study will be undertaken by means of numerical simulations in the next section.

5. Computational experiments

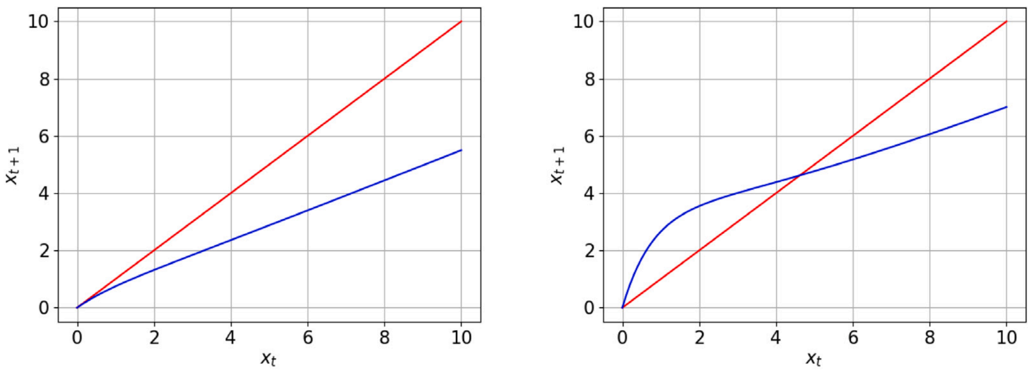
This section is dedicated to computational experiments with at least two primary objectives. First, we aim to validate the theoretical findings from the previous section. Secondly, we attempt to uncover insights not possible through analytical means. For instance, while we rigorously established that, under particular circumstances, the number of fixed points is at most three (including the origin), we could not prove this for the general case. Nevertheless, as we show next, our computational tests confirm this property more broadly.

Another goal of our computational experiments is to introduce innovative tools to explore the model's properties, such as the three-dimensional plots presented later.

Computational tests were conducted in Python, considering various parameter combinations, which from now on we refer to as scenarios. Each scenario corresponds to a particular shape of the map. Thus, the aim of these scenarios is to explore the growth behavior when the map exhibits particular properties assuming different shapes. Table 1 lists all the scenarios we analyzed, which are also depicted in Figs. 3–6.

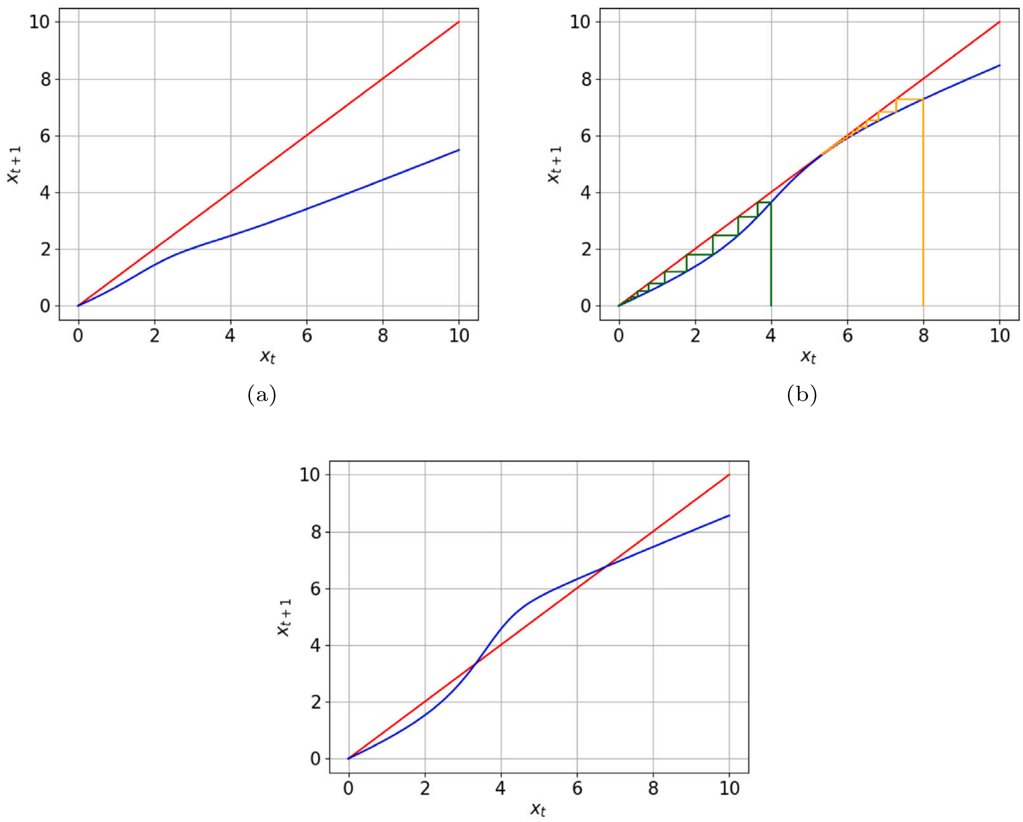
As already discussed in Proposition 3, the map can be concave when $x_c < 0$. This situation is described in scenarios 1 and 2 which are depicted in Fig. 3.

In Proposition 5, point (a), we gave sufficient conditions such that the origin is the sole equilibrium point to the map F. One of these conditions requires M to be sufficiently small, and x_c to be above a certain negative threshold. This is the case of scenario 1 represented in Fig. 3(a). If the value of M is further increased, the map F crosses the bisector of the first quadrant. According to



(a) (b)

Fig. 3. Scenario 1 (panel a) and scenario 2 (panel b).



(a) (b) (c)

Fig. 4. Scenario 3 (panel a), scenario 4 (panel b), and scenario 5 (panel c).

Proposition 5, point (b), we also have a positive equilibrium point x_1 at which the slope of the map is less than 1. In particular, the origin is a repellor, while x_1 is an attractor. This is the case of scenario 2 which is represented in Fig. 3(b).

If $x_c > 0$, the map F is not concave in principle. This is the case for the remaining scenarios. In particular, the map can be strictly increasing (scenarios 3–5) or bimodal (scenarios 6–10). The former scenarios are represented in Fig. 4.

In particular, scenario 3 in Fig. 4(a) is characterized by the condition where M is relatively small and x_c is greater than a negative threshold. As a result, according to Proposition 5, the origin is the only globally stable equilibrium point. If M is further increased, we can reach a particular limit case where the map F is tangent to the bisector of the first quadrant. This case is described by scenario 4 which is shown in Fig. 4(b). As it can also be seen from the picture, the origin is a stable equilibrium point because $|F'(0)| < 1$.

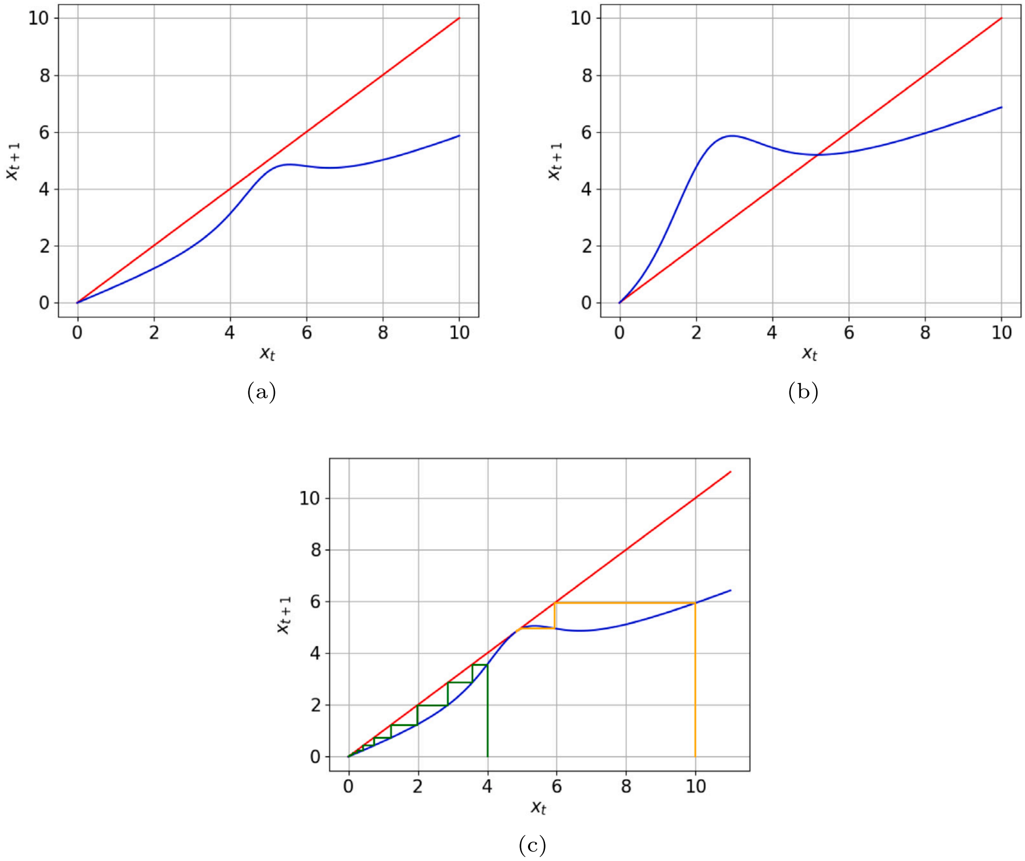


Fig. 5. Scenario 6 (panel a), scenario 7 (panel b), and scenario 8 (panel c).

The tangent point is another equilibrium point. In scenario 4 represented in Fig. 4(b), we see that this nontrivial equilibrium point has a double nature because it is unstable from the left and stable from the right. Furthermore, we observe that a tangent bifurcation occurs. In scenario 5 (depicted in Fig. 4(c)) the map is again monotonically increasing but non concave with a larger value of M. Consequently, there are two positive equilibrium points in addition to the origin. We denote these two points by x_1 and x_2, with the convention that $x_1 < x_2$. As it can be seen in Fig. 4(c), the origin is a stable equilibrium point because $|F'(0)| < 1$. x_1 is an unstable equilibrium point as $|F'(x_1)| > 1$. Finally, x_2 is a stable equilibrium point because $|F'(x_2)| < 1$. In such a case economies starting from low levels of capital per-capita may converge to the poverty trap.

The remaining scenarios are shown in Figs. 5 and 6. In these scenarios, the map F is bimodal. We name x_m and x_M the points of the local minimum and maximum of the map respectively. Evidence from the computational experiments shows that $x_M < x_m$.

In scenario 6 (depicted in Fig. 5(a)), the map exhibits a bimodal behavior, but the value of M is not sufficiently large, resulting in F always lying below the bisector of the first quadrant, except at the origin. Additionally, with $x_c > 0$ the origin emerges as the only equilibrium point, which, as per Proposition 5, is globally asymptotically stable and the economy cannot avoid the poverty trap. It emerges if M is low (i.e. if less developed economies are considered).

In scenario 7 (Fig. 5(b)), there is only a positive equilibrium point. This point is not tangent to the bisector of the first quadrant. In the particular combination of parameter values of scenario 7, we can affirm that this non-trivial equilibrium point is stable because the slope of the tangent line is less than one in modulus. Vice versa, the origin is unstable because the modulus of the tangent line is greater than one. Unfortunately, we are not able to state whether this property holds in general. In fact, this would require to tackling the equation $F(x) = x$, whose exact resolution is challenging.

A particular case is represented by scenario 8 (depicted in Fig. 5(c)), where F is bimodal with just one positive equilibrium point. However, in contrast to scenario 7, this time the map F is tangent to the bisector of the first quadrant and the tangent point coincides with the non-trivial equilibrium point. From Fig. 5(c), it turns out that this equilibrium point is stable from the right but unstable from the left. Moreover, the origin is stable.

The last case we consider is when the map F crosses the bisector of the first quadrant at the origin and at two positive equilibrium points (say x_1 and x_2 with $x_1 < x_2$). This situation is represented by scenarios 9 and 10 reported in Fig. 6.

As it can be seen from Figs. 6(a) and 6(b), in these two last scenarios the origin is stable and x_1 is unstable. In fact, any starting point $x_0 \in (0, x_1)$ generates a sequence approaching 0 as $t \to +\infty$. Vice versa, any starting point such that $x_0 > x_1$ and $x_0 \neq x_2$

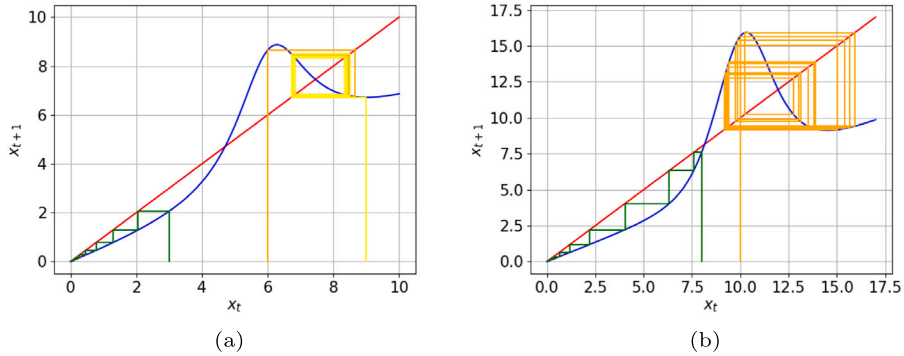


Fig. 6. Scenario 9 (panel a) and scenario 10 (panel b).

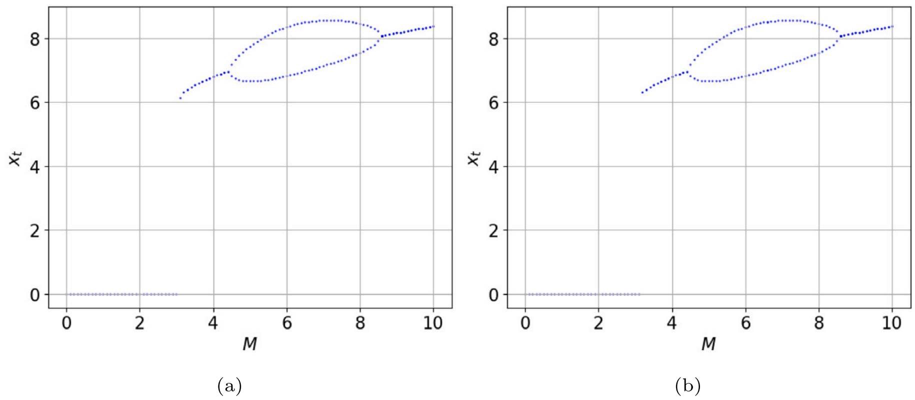


Fig. 7. M-bifurcation diagram for scenario 9 starting from x_M (panel a) and x_m (panel b).

generates a periodic or eventually periodic sequence. These results open the door to chaotic analysis. We conducted this analysis starting from scenario 9, but mainly concentrating our numerical efforts on scenario 10 for a reason that will be explained soon.

As already stated, our model is characterized by a considerable number of parameters. Nevertheless, an analysis where all the parameters can vary is not manageable, for at least two reasons. First, the number of diagrams resulting by considering all the combinations of parameters would be too large. Secondly, it is not possible to represent diagrams with a dimension greater than three. For these reasons, we decided to investigate the chaotic behavior by varying the three main parameters of the model, while keeping constant the remaining ones. These parameters are M, x_c, and κ. The importance of these parameters is confirmed by the following considerations: first, M is particularly significant in terms of the peak of a possible local maximum. The larger M is, the higher this peak. Moreover, as already proved and as already seen in scenarios 1, 3, and 6, if M is set small enough, then F has no equilibrium points but the origin. Said in simpler words, M resembles the amplitude of a sinusoidal function $y = A\sin(\omega t + \phi)$. Depending on the value of A, the peaks of the sinusoidal change accordingly.

Secondly, x_c is the inflection point of the σ_κ function. Although x_c is not (in principle) an inflection point of F, it significantly influences the map.

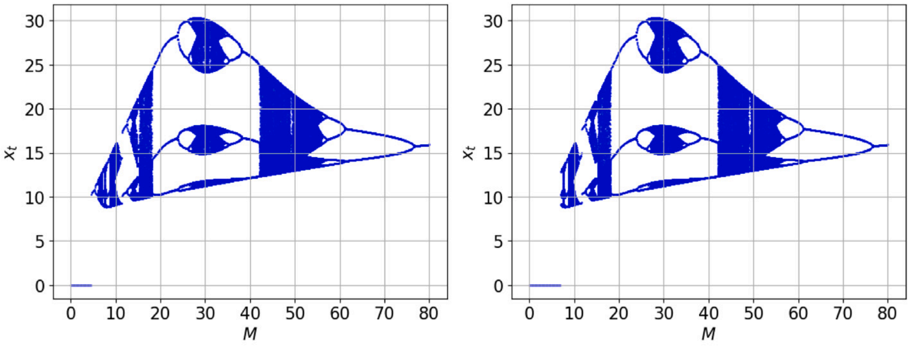
Finally, as we showed in Figs. 1 and 2, the parameter κ influences the shape of the $\exp_\kappa$ and σ_κ functions. Consequently, it also influences the shape of the map.

Dealing with three varying parameters, there is an abundant number of ways we can combine them to study chaotic behavior. In particular, we propose 1D, 2D, and 3D bifurcation diagrams. Concerning 1D graphs, we will show bifurcation diagrams by varying in turn M, x_c, and κ. For 2D graphs, we vary these parameters in pairs. Therefore, we have the following combinations: M vs. x_c, M vs. κ, and x_c vs. κ. Having three varying parameters, we only have one 3D graph.

We consider the dynamics starting from the maximum point x_M and the minimum point x_m.

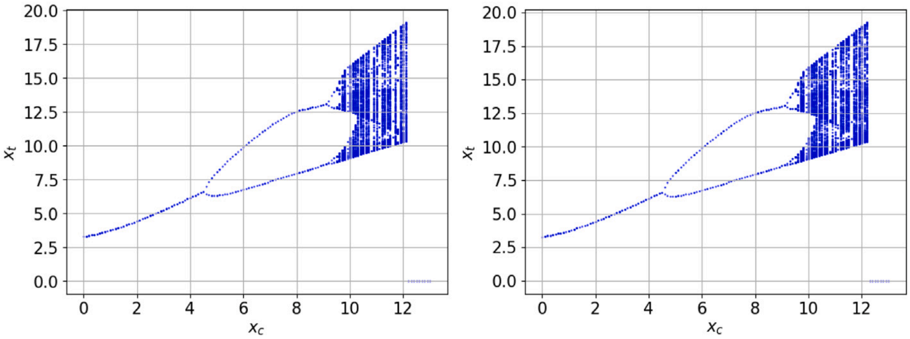
Our first bifurcation diagrams are shown in Fig. 7, where we vary M from 0 to 10 and setting the other values of the parameters to those of scenario 9. In Fig. 7(a), we start from x_M, while in Fig. 7(b) we start from x_m.

The two graphs are pretty similar. In particular, we can observe that for values of M below a threshold around 3, the origin is the only attractor. This is consistent with Proposition 5, as for sufficiently small values of M and x_c exceeding a negative threshold, the origin acts as a globally asymptotically stable equilibrium point i.e. less developed economies may converge to the poverty trap. For higher values of M, the origin becomes unstable and at least a positive equilibrium point exists. This justifies the discontinuity



(a) (b)

Fig. 8. M-bifurcation diagram for scenario 10 starting from x_M (panel a) and x_m (panel b).



(a) (b)

Fig. 9. x_c-bifurcation diagram for scenario 10 starting from x_M (panel a) and x_m (panel b).

of the bifurcation diagram around 3. We also observe a bifurcation occurring for a value of M between 4 and 5. Moreover, the steady state is not attractive for $M \in (4.5, 8.5)$ and the dynamics converge toward an attractive period-two cycle. These dynamics show the presence of a cycle but are not chaotic. For this reason, we conducted similar analyses, this time considering scenario 10. From these analyses, it turned out that the dynamic system can have chaotic behavior. This can be clearly observed in Figs. 8(a) and 8(b) where, setting the parameter values to those of scenario 10, we varied M between 0 and 80.

Both bifurcation diagrams show different behaviors according to the value of M. As expected, below a particular threshold, the origin is a global attractor. After this threshold, we observe in turn periodic, chaotic, and converging dynamics, thus confirming that developed economies can grow over time and even if cycles can emerge.

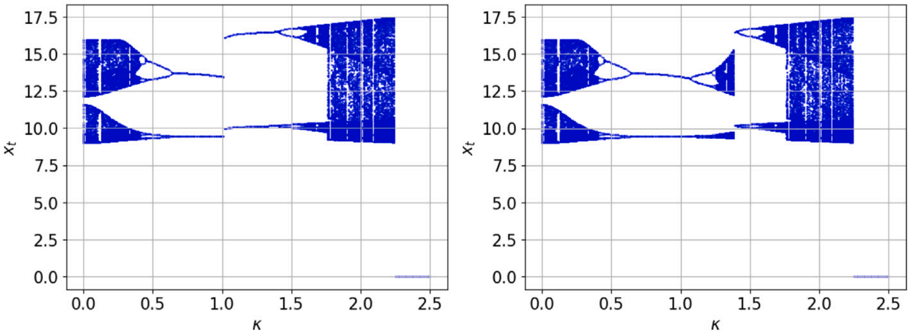
Similar diagrams are depicted in Figs. 9 and 10, where we respectively vary x_c and κ, while keeping the other parameters fixed to the values of scenario 10. Such figures confirm that fluctuations may arise with convex–concave production functions and that when rare events are very likely to occur, the poverty trap may emerge.

So far, we have presented bifurcation diagrams where only one parameter varies. However, a similar analysis with two parameters varying simultaneously can be undertaken. In particular, three combinations are possible. The first combination consists of varying the parameters M and x_c. The related diagrams are depicted in Fig. 11. In these diagrams, there is a color associated to a specific coordinate, say (M', x_c'). This color is related to the period of the orbit starting from x_M in Fig. 11(a) and from x_m in Fig. 11(b), with $M = M'$, $x_c = x_c'$, and the remaining parameters set to the values of scenario 10. The association of the color to the period is shown in the legends in Figs. 11(a) and 11(b).

The other combinations with two parameters varying at the same time are M with κ and x_c with κ. The diagrams of these combinations are respectively reported in Figs. 12 and 13.

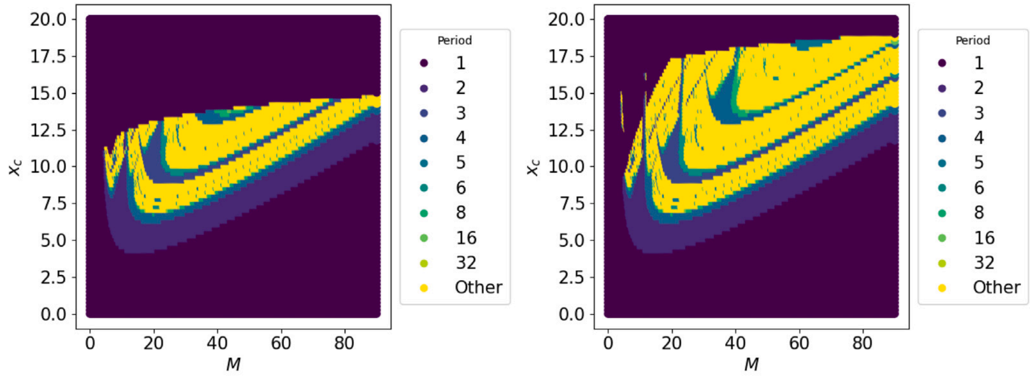
The last plots we propose are about the combination of all three parameters M, x_c, and κ. These plots are shown in Figs. 14, 15, and 16. We would like to emphasize how difficult it is to appreciate a three-dimensional plot presented in an article as it is not possible to explore what happens inside it. For this reason, we have decided to make available the code used to create the plots of Figs. 14, 15, and 16.[1] The plots were created using the Plotly Python package. One remarkable aspect of Plotly is that once the code

[1] In particular, the interested reader can find the code in *The κ-logistic growth repository*, https://github.com/maurobaldi/kLogisticGrowth.



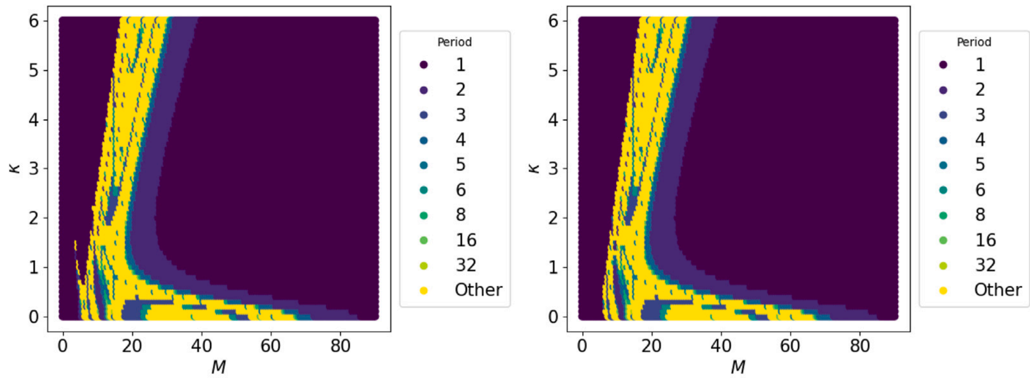
(a) (b)

Fig. 10. κ-bifurcation diagram for scenario 10 starting from x_M (panel a) and x_m (panel b).



(a) (b)

Fig. 11. Two-dimensional diagram for scenario 10 with M and x_c on the axes, starting from x_M (panel a) and x_m (panel b). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) (b)

Fig. 12. Two-dimensional diagram for scenario 10 with M and κ on the axes, starting from x_M (panel a) and x_m (panel b).

is executed, the plot appears in a browser window, where it is possible not only to zoom in but also to rotate it and explore areas of interest that may not be visible from a simple two-dimensional view, i.e., how the graph would appear on a sheet of paper. For example, Figs. 14 and 15 show a slice of the associated graph corresponding to x_m and x_M, respectively. Fig. 15 also demonstrates another capability of Plotly: by hovering the mouse cursor over a point of interest, the information associated with that point is displayed next to the cursor. Finally, Fig. 16 is an enlarged and internal view of the graph in Fig. 15.

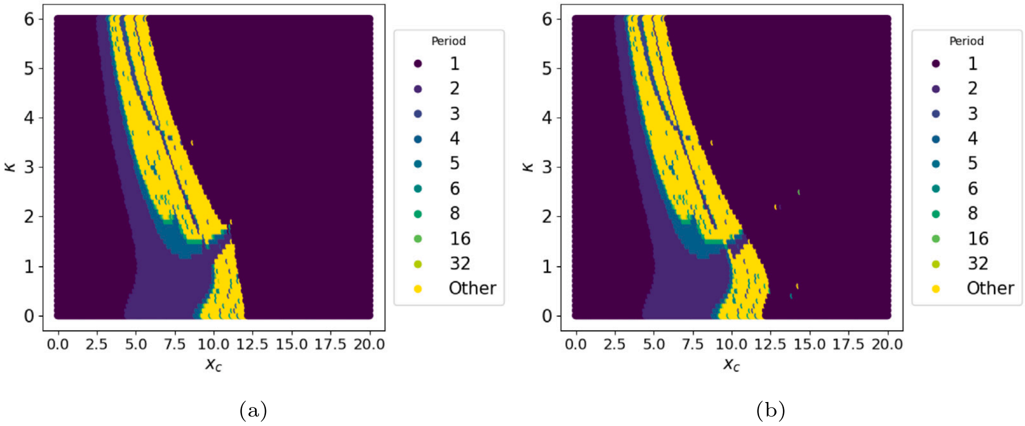


Fig. 13. Two-dimensional diagram for scenario 10 with x_c and κ on the axes, starting from x_M (panel a) and x_m (panel b).

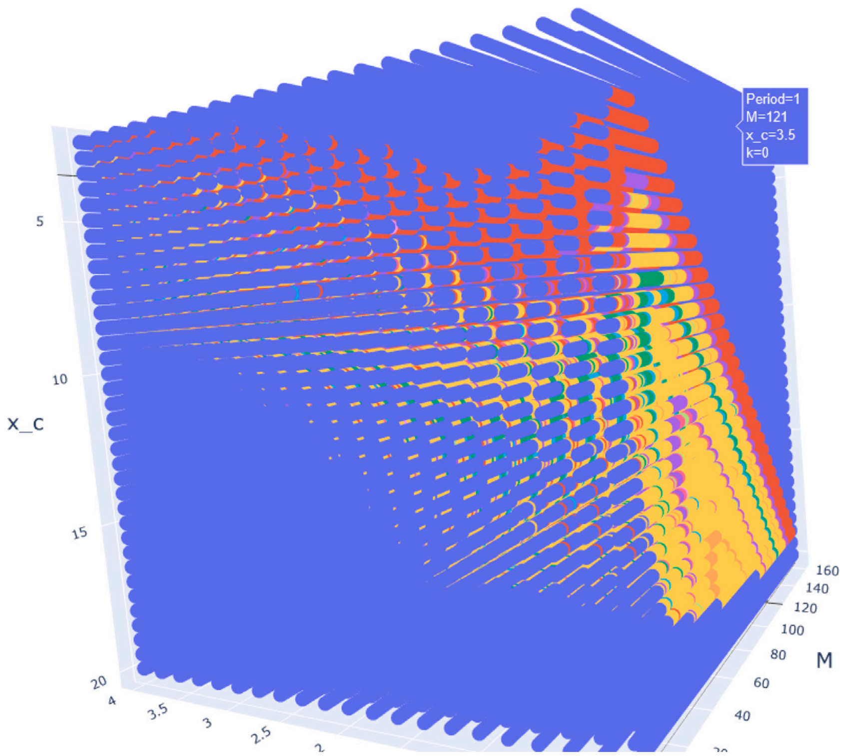


Fig. 14. A particular three-dimensional view in Plotly associated with x_m.

6. Conclusions

In this paper we have explored a growth model using the κ-logistic function coming from a modified exponential function able to consider the occurrence of rare events.

The resulting production function leads to a generalization of the sigmoidal function, whose growth model has already been studied in existing literature. This approach offers several advantages, including the ability to handle both concave and non-concave production functions and to consider economies at different development levels or passing through rare events. As demonstrated, multistability emerges, meaning that both the origin and an alternative attractor, even complex, may be stable. Consequently, different initial conditions might result in economies displaying diverse growth patterns. Our theoretical results were substantiated by comprehensive computational experiments that showed how the dynamical system behaves under all the most significant circumstances.

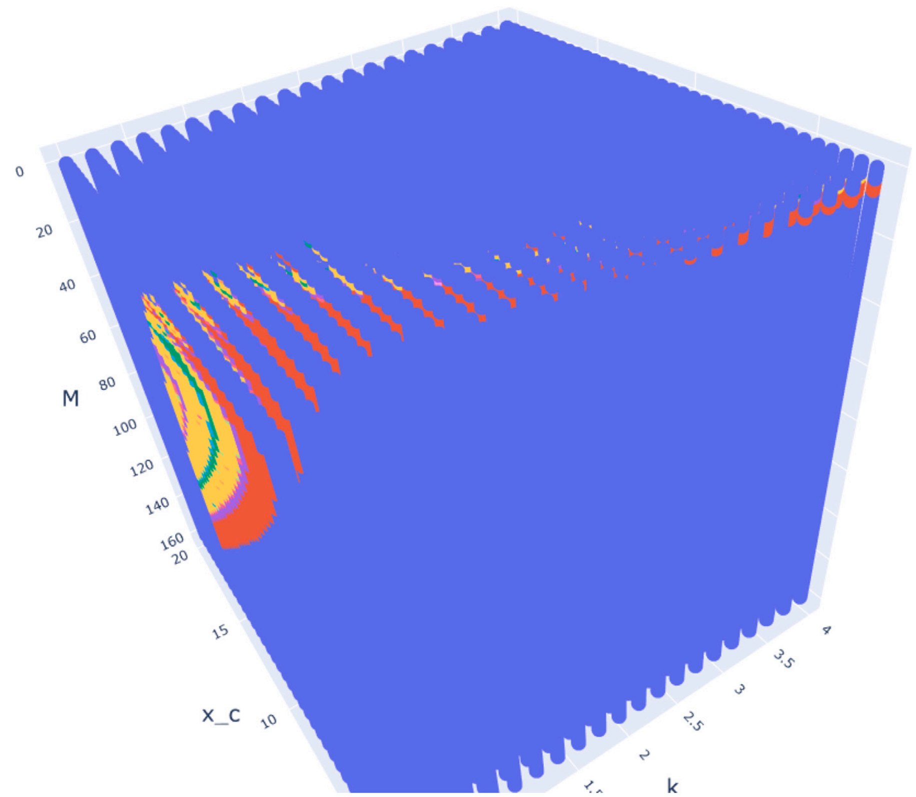


Fig. 15. A particular three-dimensional view in Plotly associated with x_M.

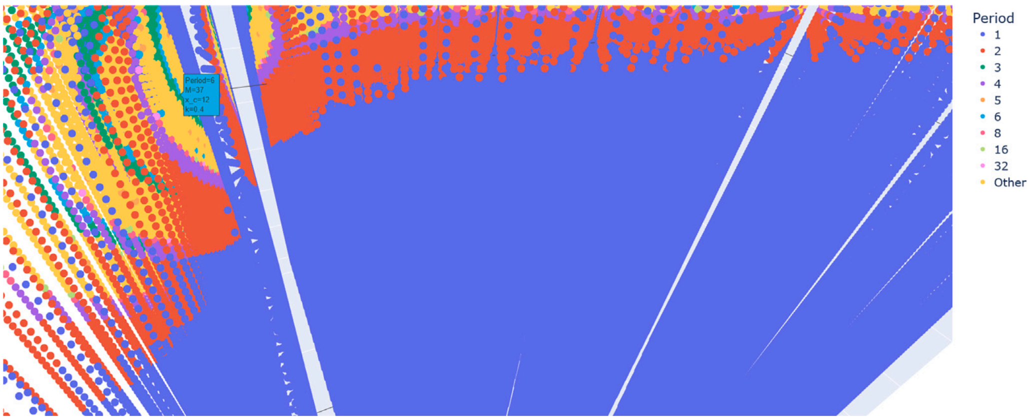


Fig. 16. A three-dimensional zoomed view in Plotly associated with x_M.

Our findings show that with S-shape production functions, economies at low development level or experiencing a sufficiently high probability to incur in rare events can fail in the poverty trap. Differently, if the economy is sufficiently developed, then concave production functions produce economic growth patterns converging to the unique positive steady state even in the presence of relevant rare events.

Encouraged by these outcomes, we plan to pursue future developments. Potential directions include a more in-depth exploration of the basins of attractions or the application of the κ-logistic function in multi-dimensional growth models.

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CRediT authorship contribution statement

Mauro Maria Baldi: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Cristiana Mammana:** Writing – review & editing, Writing – original draft, Methodology, Conceptualization. **Elisabetta Michetti:** Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the revision of this work, the authors used ChatGPT in order to proofread most of the paper. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication. Additionally, once the work was completed, the article was proofread by a native English speaker.

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