CERME 13

13TH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

10-14 July 2023
Budapest
Hungary

PROCEEDINGS OF THE THIRTEENTH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

Editors: Paul Drijvers, Csaba Csapodi, Hanna Palmér, Katalin Gosztonyi and Eszter Kónya

Organised by: Alfréd Rényi Institute of Mathematics and Eötvös Loránd University
Budapest, Hungary

2023

erme
european society for research in mathematics education

Alfréd Rényi Institute of Mathematics

Eötvös Loránd University Faculty of Science

Editors

Paul Drijvers[1], Csaba Csapodi[2,3], Hanna Palmér[4], Katalin Gosztonyi[2,5] and Eszter Kónya[6]

[1]Utrecht University, Utrecht, the Netherlands
[2]Eötvös Loránd University, Budapest, Hungary
[3]Alfréd Rényi Institute of Mathematics, Budapest, Hungary
[4]Linnaeus University, Växjö, Sweden
[5]University of Montpellier, Montpellier, France
[6]University of Debrecen, Debrecen, Hungary

Publisher

Alfréd Rényi Institute of Mathematics, Budapest, Hungary and ERME
ISBN 978-963-7031-04-5

Towards relational thinking by Matlab LiveScript in linear algebra

Giovannina Albano[1], Angela Donatiello[1] and Agnese Ilaria Telloni[2]

[1]University of Salerno, Italy; galbano@unisa.it

[2]University of Ancona, Italy

This paper is focused on the use of Matlab LiveScript in linear algebra courses for freshmen, aimed at fostering a transition from an instrumental approach to a more relational one when students solve a mathematical task. To analyse students' explanatory writing of a mathematical solution text we choose a model which looks at the ATD frame, with the addition of a logical chain component, from the point of view of Grice's maxims. The preliminary results show a transition from an instrumental approach to a relational approach, emerged from improved verbal arguments that make evident the relationships among the various mathematical objects.

Keywords: University, Matlab, linear algebra, instrumental/relational approach.

Introduction

This paper wants to contribute to the debate on the opportunities that can be offered using domain-specific software in teaching mathematics to engineering students. According to Kanwal (2018), we underline that the engineers' professional activities make use of technology for solving mathematical tasks and this should be considered from an educational point of view. This has been our assumption for introducing the use of mathematical software in the basic mathematics courses we teach. In particular, we choose Matlab since it is the most popular mathematical software used by engineers and it is particularly suitable for a Linear Algebra course. Lavicza (2008) also conducted a study on the positive influence and potential that CAS could have as a useful mathematical tool for students' studies and careers, highlighting both the benefits and resistance to introducing CAS in undergraduate courses. The main uses of CAS at university level reported in literature (Lavicza, 2010) concerns the possibility of visualise mathematical concepts in a lecture setting as well as of engaging students in experimentation and solving real word problems. Sometimes, teachers use CAS 'behind the scenes' to prepare and assign homework, checking the solutions. In contrast, a smaller percentage of teachers integrate CAS in final exams. Our didactical objective in introducing the use of Matlab was to promote a transition from a hard instrumental approach to a more and more relational one (Skemp, 1976), when students solve a mathematical task. The didactical approach we adopted in using Matlab is in line with Buchberger's (1990) idea of moving from a "white-box" phase to a "black-box" phase. The first one corresponds to the phase when a mathematical object is introduced to the students and they need to learn both the conceptual aspects and the procedural/computational aspects. The "black-box" phase can come into play later when the students master the mathematical object, and they can manage the procedure as an object itself, without going into the details of the computations. Especially, we chose to use Matlab LiveScript which allows the users to combine verbal text and code to be run. The use of CAS as a "black-box" can be seen as a model that acts as meta-informational knowledge to promote and evaluate reflection on an overly complex system and thus generate new knowledge (Greubel & Siller, 2022). Our research hypothesis is that the use of Matlab LiveScript for moving from "white-box" to "black-box" can foster a transition from an instrumental approach to a relational approach: the chance of calling procedures as "black-box" can promote attention to the verbal comments, that can be used to add (possibly theoretical) arguments supporting the solving plan, and a change in the quality of the communication of such arguments.

Theoretical framework

In this study, we will focus on the students' transition from an instrumental approach to a relational one when they pass from a traditional paper and pencil task to a task to be solved by the use of Matlab. In line with Skemp (1976), we can say that in an instrumental approach to knowledge, students start from some predetermined starting points to achieve predetermined goals through rules and usage skills, without reasoning. In contrast, in a relational approach to knowledge, students construct conceptual structures through which achieve goals, actively exploring new areas. A further component of our theoretical framework is the Anthropological Theory of the Didactic (ATD; Chevallard, 1992). ATD proposes a model for human activities, used also for mathematical ones, based on the notion of *praxeology* (or mathematical organisation). It is composed of four components, grouped into two clusters: *praxis* and *logos* (Bosch & Gascon, 2014). *Praxis* is composed of a type of task and suitable techniques to solve this task. It represents the "know-how", that is the set of practical components. *Logos* is composed of a technology, namely a discourse on the technique that explains and warrants the techniques, and a theory, which is a discourse to justify the technology. It represents two levels of justification and description of the *praxis*. Being interested in analysing students' written communications when they solve a task, Albano et al. (2023) introduced a further component, consisting of the logical chain present in the written protocols, besides the three ATD components. This component refers to a *logical chain of deduction,* which is a logical chain of arguments supporting the solution provided by the student. It can be found in the explanatory writing of the mathematical text and the logical grammatical structure of a sentence. Moreover, for taking in more account the communication feature of the protocols, in their model Albano et al. (2023) looked at the extended ATD components from the point of view given by Grice's (1975) four maxims, i.e., Quantity (how much information), Quality (true information or supported by evidence), Relation (relevance of the information for the topic) and Manner (language used to make information understandable). The authors also introduced various levels of coding for each component (Figure 1). Quantity is coded as poor, enough, or over, depending on how much information is explicitly given and needed to understand the solution, and not left to the reader to be interpreted. Quality is coded as good, fair, and bad, depending on the correctness of the solution and the demonstration of adequate evidence. Relation is coded as relevant or irrelevant to the task, depending on the reference or not of the student's solution to the context of the task. Manner is coded as clear, ambiguous, or obscure, depending on the clearness of the solution provided by the student and the need of being interpreted to be understood.

	Elaborated ATD components			
Grice maxims	Technique	Technology	Theory	Logical chain
Quantity (Over/Enough/Poor)				
Quality (Good/Fair/Bad)				
Relation (Relevant/Irrelevant)				
Manner (Clear/Ambiguous/Obscure)				

Figure 1. The model for analysing students' communication

Methods

The study involved around 60 Computer Engineering freshmen, attending a course in Linear Algebra. The course developed over 12 weeks, with three face-to-face 2-hour classes (both lectures and exercises sessions) per week. The course provides the students with tutoring face-to-face sessions (2 hours per week), didactical material, and resources available on the platform Moodle, such as videos, notes from the digital board, slides, weekly tasks, and quizzes. This study follows a qualitative research method and focuses on the evolution of the students' praxeologies when they solve three linear algebra tasks. The first two tasks have been assigned as homework, during the same week of the course. The third task is part of the exam taken just at the end of the course. All the tasks explicitly require the student to give reasons for their answers. In the following, the specific request of each task (see Figure 2 for the matrices the tasks refer to):

Task 1: Compute the eigenvalues of matrix A and the geometric and algebraic multiplicity of each eigenvalue.

Task 2: Establish if the matrix B is diagonalisable over $\mathbb{R}$ and $\mathbb{C}$.

Task 3: Establish if the matrix C is diagonalisable over $\mathbb{R}$ and $\mathbb{C}$.

$$A = \begin{pmatrix} -1 & -2 & 2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \quad \mathsf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \quad \mathsf{C} = \begin{pmatrix} 0 & -15 & 120 & -60 \\ -90 & 15 & -360 & 180 \\ -18 & -6 & -12 & 6 \\ -36 & -12 & 36 & -18 \end{pmatrix}$$

Figure 2. The matrices involved in the three tasks

The different size of the elements in the two matrices B and C, used in the tasks solved by Matlab, depends on the fact that B appeared in Task 2, when the students are at the beginning of the use of Matlab, whilst C appeared in Task 3, given at the end of the course. Moreover, the third matrix is characterised by elements that make manual calculations more difficult. We analysed the protocols the students produced to solve the three tasks. All the tasks concerned the diagonalisation of a matrix, but they slightly differ in the specific requests and the solving setting. The first task was paper and pencil homework, the second task is given as homework to be performed producing a LiveScript Matlab file, just like the third one, which was to be solved in the exam setting. The analysis of each protocol has been performed according to the model shown in Figure 1. We specify that in our case we consider two facets of "over" for what concerns the Quantity: on the one hand we mean useless information to achieve the aim (here 'over' has a negative meaning), and, on the other hand, we interpret "over" as additional information necessary to better explain the process (here 'over' has a positive meaning).

Analysis and preliminary findings

In this section, we show the analysis of two paradigmatic cases, related to the students S1 and S2, chosen because S1 can be considered a low-medium achiever whilst S2 is a good achiever, with respect to the mark obtained at the final exam.

The case of S1: Let us look at the protocol shown in Figure 3, corresponding to the solution provided by S1 performed as homework related to Task 1. S1 uses a clear technique, with bad quality (i.e., incorrect invariance of the determinant upon any elementary row operations), relevant to the object. The quantity of the technique appears over: indeed, to compute the geometric multiplicity of the eigenvalue -3, S1 performs computations of the rank of the matrix $A + 3I$ (technique), justifying it by citing the formula $m_g = n - rk(A + 3I)$ (technology), although she recalls the inequality $1 \leq m_g \leq m_a$ (theory). Although such a technique is in general correct, the theory and the logical chain failed, as the theory is not used to deduce some conclusion (assuming its truth), but it has been regarded as something to be verified. The manner appears globally to be ambiguous: in particular, the reader should interpret how the two ‘implications’ (first row and last row) are connected.

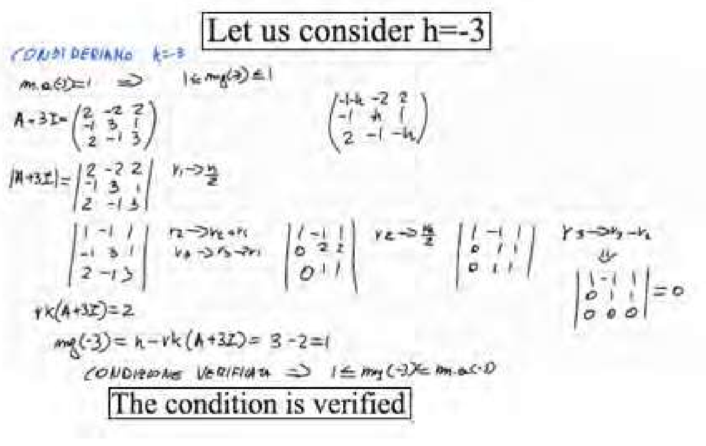

Figure 3. Task 1 of the student S1

Let us move to analyse the protocol in Figures 4 and 5, which shows the solution to Task 2 provided by S1 as homework. We can see that S1 uses the Matlab function to compute the eigenvalues (technique) and she improves in applying correctly the theory and the logical chain: indeed, she recalls the same inequality used in Task 1, but in this case, it is used to deduce the value of the geometric multiplicity, without performing useless computations (i.e., rank). The logical chain concerning the inequality is used correctly and made evident by the construct “if…, then…”. Nonetheless, the manner remains globally ambiguous (for instance, ‘if we sum the eigenspaces’) and sometimes obscure (i.e., the connection between the main characterisation of diagonalisation and the dimension of the eigenspace is not clear).

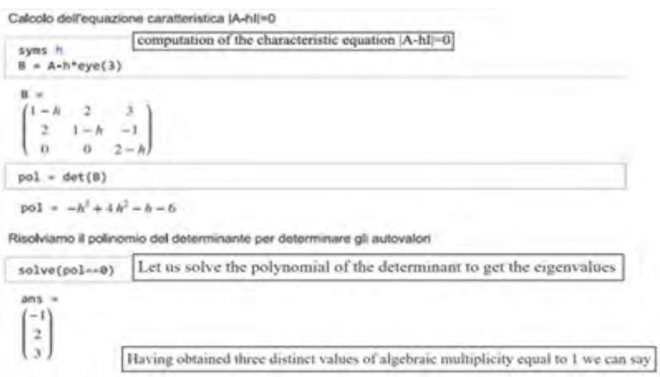

Figure 4. Task 2 of the student S1 – part 1

Avendo ottenuto tre valori distinti di molteplicità algebrica pari a 1, possiamo dire

Punto a)

La matrice A è sicuramente diagonalizzabile perchè se 1<= mg(h)<= ma(h) allora se 1<=mg(h)<=1 la molt. geometrica di tutti gli autovalori è sicuramente pari a 1 e per la caratterizzazione principale delle matrici diagonalizzabili sappiamo che la dimensione di un autospazio è dato dal valore della sua molt. geometrica, quindi se sommiamo gli autospazi avremo esattamente la dimensione dello spazio R^3

> The matrix A is definitely diagonalizable because if 1<=mg(h)<=ma(h) then if 1<=mg(h)<=1 the geometric molt. of all the eigenvalues is definitely equal to 1 and by the main characterization of the diagonalizable matrices we know that the dimension of an eigenspace is given by the value of its geometric multiplicity, therefore if we sum the eigenspaces we will have exactly the dimension of the space R^3

Figure 5. Task 2 of the student S1 – part 2

Moving to Task 3, let us analyse the protocol in Figure 6, i.e., the solution provided by S1 when she is in the exam context. We note an improvement in the quality and quantity related to the theory and the logical chain. Indeed, the discourse on technology is enriched by more references to theorems and definitions, also distinguishing between the two kinds of mathematical constructs (i.e., multiplicity). The manner of the theory and the logical chain appears clear, with a residual excerpt when the student writes "the geometric multiplicity […] is equal to $1 \leq mg(h) \leq ma(h)$ then for each eigenvalue."; actually, she means "since $1 \leq mg(h) \leq ma(h)$ then the geometric multiplicity is…".

```
pol = det(B)
```

pol = $h^4 + 4h^3 - 59h^2 - 126h + 720$

```
solve(pol==0,h)
```

ans =

$$\begin{pmatrix} -8 \\ -5 \\ 3 \\ 6 \end{pmatrix}$$

> We have found 4 distinct eigenvalues whose algebraic multiplicity is equal to 1 for all, knowing that the geometric multiplicity, which would be the dimension of the eigenspace associated to the eigenvalue, is equal to 1<=mg(h)<=ma(h) then for each single eigenvalue the geometric multiplicity will be 1, summing the mg of all the eigenvalues we can find the dimension of the whole space R^4 therefore by the main characterization of the diagonalizable matrices the matrix A is diagonalizable in R

Abbiamo trovato 4 autovalori distinti la cui molteplicità algebrica è pari ad 1 per tutti, sapendo che la molteplicità geometrica, che sarebbe la dimensione dell'autospazio associato all'autovalore, è pari a 1<=mg(h)<=ma(h) allora per ogni singolo autovalore la molteplicità geometrica sarà 1, sommando le mg di tutti gli autovalori possiamo trovare la dimensione dello spazio ambiente R4 quindi per la caratterizzazione principale delle matrici diagonalizzabili la matrice A è diagonalizzabile in R, ed essendo l'insieme dei numeri reali R a sua volta sottoinsieme del piano complesso C allora la matrice sarà diagonalizzabile anche in quest'ultimo.

Figure 6. Task 3 of the student S1

The case of S2: Let us analyse the protocol in Figure 7, showing the solution to Task 1 provided by S2 as homework. The student S2, although she uses enough technique whose quality can be recognised as fair, the manner is ambiguous as it is left to the reader's interpretation. The technology is not explicitly communicated, because we can see the lack of any justification (even symbolically) for the technique chosen. The theory is of bad quality, because she writes the relationship $1 \leq m_g(h) \leq m_a(h)$, but does not use it when it's necessary, always implementing a useless procedure. There is no evidence of logical chains.

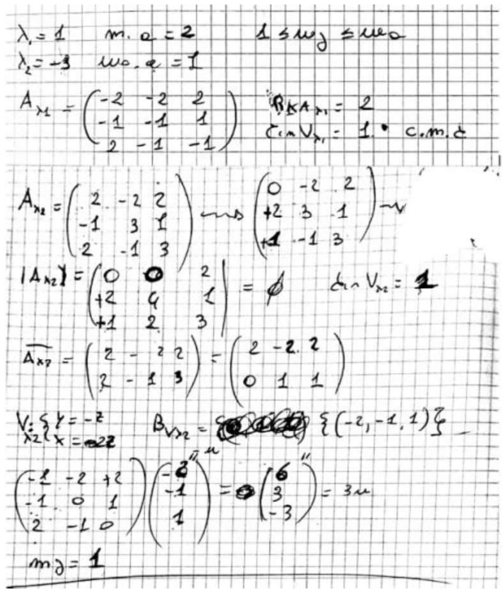

Figure 7. Task 1 of the student S2

Let us move to the protocol in Figure 8, showing the solution to Task 2 provided by S2 as homework.

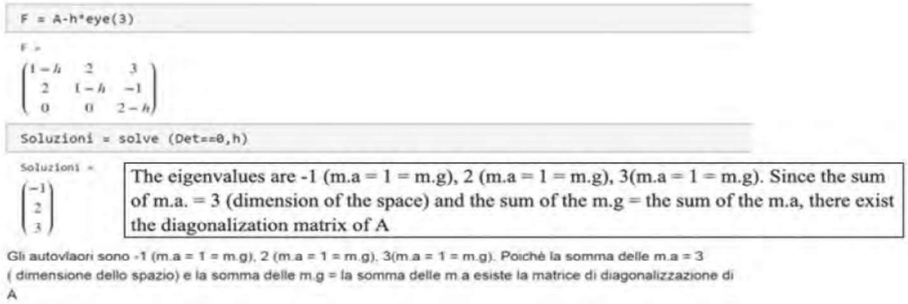

Figure 8. Task 2 of the student S2

The technology used by S2 is supported by a theory referred sometimes in an ambiguous manner (e.g., $m.a = 1 = m.g$) and sometimes in an obscure manner (e.g., 'since the sum of $m.a. = 3$ […] and the sum of the $m.g. =$ the sum of the $m.a.$'). The latter remark gives rise to the bad quality of the logical chain. Moving to Task 3, let us analyse the protocol in Figure 9, which is the solution provided by S2 when she is in the examination context. In Task 3, we note the improvement of the discourse about the multiplicities: here the reference to the theory appears in a clear manner and of a good quality (e.g., the inequality…), and analogously for the logical chain (e.g., the correct use of the theory to deduce the geometric multiplicity).

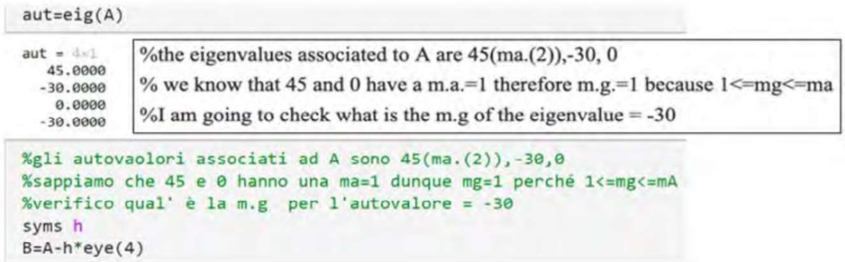

Figure 9. Task 3 of the student S2 - part 1

The reference to the theory becomes clearer and the quality of the logical chain becomes good (e.g., ‘thus’, given that’, …) (Figure 10).

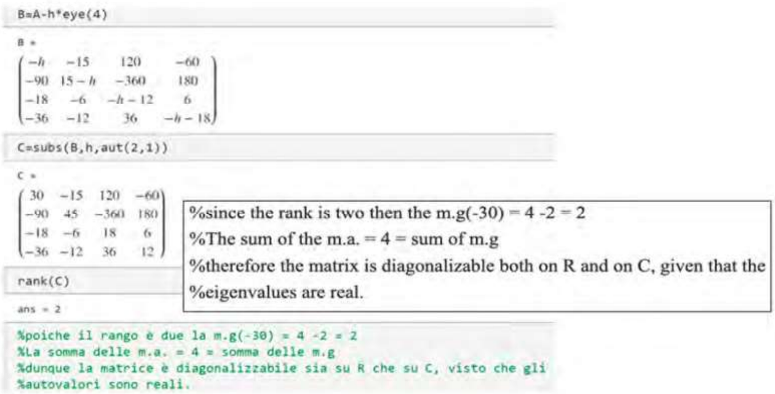

Figure 10. Task 3 of the student S2 - part 2

Finally, it is worth to note that S2 uses the Matlab function eig(A) (as already done in Task 2), adding as comments the alternative functions to be used corresponding to a paper and pencil procedure to reach the same final output, thus making the function eig explicit and even clearer to the reader. This shows a process of packing and unpacking, that is of moving from black-box to white-box (Figure 11).

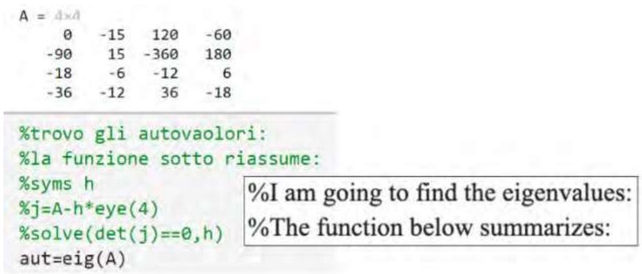

Figure 11. Task 3 of the student S2 - part 3

Discussion and conclusions

With respect to our research hypothesis, the data analysis shows a change in the students’ protocols, highlighting a transition from a purely instrumental approach, shown by the focus on the procedures, to a relational approach, emerged by improvement in verbal arguments, showing that make evident the relationships among the various mathematical objects. In particular, the transfer of the computations to Matlab functions, used as “black-box”, allowed the students to focus on the theory that supports the solving process more than on the technique. Furthermore, the quality of communication changed, as there has been an improvement in both verbal comments and the deductive logical connections needed to argue. This corresponds, in the ATD framework, to the emergence of the components of the theory and the logical chain. From Grice’s (1975) viewpoint, we also note an evident improvement in the manner of communication, which changed from obscure or ambiguous to clearer one. Last but not least, some students showed to be conscious of the techniques underlying a more complex function (e.g., eig(A)), adding comments consisting in unpacking the used function and showing a corresponding sequence of lower-level commands. It seems to make evident the students' deeper level of mastering the mathematical object. The preliminary findings encourage us to deepen the study presented in this paper, exploring the relationship between the improvement from the communicational point of view and the deeper comprehension of the mathematical objects, which are the two dimensions of the model in Figure 1. Moreover, coming back to paper and pencil can give some further data on the internalisation of such improvement and comprehension. Further research could concern the integration of the instrumental orchestration.

References

Albano, G., Swidan, O., & Pierri, A. (2023). A model for analyzing the explanatory writing of undergraduate students when solving mathematical tasks. *International Journal of Mathematical Education in Science and Technology, 54*(2), 180–194, https://doi.org/10.1080/0020739X.2021.1949757.

Bosch, M., & Gascón, J. (2014). Introduction to the Anthropological Theory of the Didactic (ATD). In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in Mathematics education* (pp. 67–83). Springer. https://doi.org/10.1007/978-3-319-05389-9_5.

Buchberger, B. (1990). Should Students Learn Integration Rules? *ACM SIGSAM Bulletin, 24*(1), 10–17, https://doi.org/10.1145/382276.1095228.

Chevallard, Y. (1992). Fundamental concepts in didactics: perspectives provided by an anthropological approach. In R. Douady & A. Mercier (Eds.), *Research in Didactique of mathematics, selected papers* (pp. 131–167). La Pensée sauvage.

Greubel, A., & Siller, H. S. (2022). Learning about black-boxes: A mathematical-technological model. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12).* (pp. 2741–2748). ERME / Free University of Bozen-Bolzano. https://hal.science/hal-03748388.

Grice, H. P. (1975). Logic and conversation. In P. Cole & J. Morgan (Eds.), *Syntax and semantics, Vol. 3, Speech Acts* (pp. 41–58). Academic Press.

Kanwal, S. (2018). Engineering students' engagement with resources in an online learning environment. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M Hogstad (Eds.). *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2018)* (pp. 145–154) University of Agder and INDRUM. https://hal.science/hal-01849939.

Lavicza, Z. (2008). Factors influencing the integration of computer algebra systems into university-level mathematics education. *International Journal for Technology in Mathematics Education, 14*(3), 121–129.

Lavicza, Z. (2010). Integrating technology into mathematics teaching at the university level. *ZDM–The International Journal on Mathematics Education, 42*(1), 105–119, https://doi.org/10.1007/s11858-009-0225-1.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.