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ABSTRACT: Since the beginning of the twentieth century, prominent authors including Jean Piaget drew attention to Edmond Goblot's account of mathematical thought experiments. But his contribution to today's debate has been neglected so far. The main goal of this paper is to reconstruct and discuss Edmond Goblot's account of "logical operations" (the term he used for thought experiments in mathematics) and its interpretation by Piaget against the theoretical background of two open questions in today's debate: 1) the relationship between empirical and mathematical thought experiments, and 2) the question of whether mathematical thought experiments can play a justificatory function in proofs. The main corollary of this analysis is that Piaget's interpretation is seriously flawed and insufficiently appreciative of important theses of Goblot's account. First, Goblot can be easily defended against Piaget's main criticism, and secondly, Goblot developed ideas about mathematical thought experiments that still deserve attention.

KEYWORDS: mathematical thought experiments; mathematical vs empirical knowledge; psychology vs logic of knowledge; Goblot Edmond; Piaget Jean; Rignano Eugenio; Chaslin Philippe.

1. Introduction

"It was Mach [...] who first discovered that reasoning is a thought experiment [*expérience pensée*], a *Gedankenexperiment*. Eugenio Rignano [...] generalized and developed this conception and dealt with mathematical reasoning for a long time. Goblot, for his part, arrived at the same result [...] but in another form, the so-called logical form, to which he was indeed obliged to add the operation, which has nothing to do with what we usually call logic." (Chaslin 1926, 209)

By these few lines Chaslin tried to sketch in an extremely condensed form the first history of philosophy of mathematical thought experiments (hereafter MTE). Prominent authors such as André Lalande and Jean Piaget would later call attention not only to Eugenio Rignano[1] and Edmond Goblot[2], but also to Philippe Chaslin[3], and discuss their main theses. However, these authors have been forgotten, and are not now referenced in the current debate on the epistemological and methodological status of TEs.

The neglect of these authors would be worth explaining since two well-known authors, Lalande and Piaget, discussed their ideas in well-known works. Lalande not only called attention to Goblot in 1919a and 1919b, but also in his famous and commonly used *Vocabulaire technique et critique de la philosophie* (1932), where he included the entry "Expérimentation mentale" (French translation of Ernst Mach's *Gedankenexperiment*). With special reference to the first chapter of "Rignano, *Psychologie du raisonnement*" (Rignano 1920) and to the eleventh chapter of "Goblot, *Traité de logique*" (Goblot 1918), he notes that "[t]his expression, and, moreover, the analysis of the operation which it designates, have become very usual." (Lalande 1932, Tome III). Piaget drew attention to and discussed

Edmond Goblot, Eugenio Rignano and Philippe Chaslin's accounts of TEs beginning in 1922, in direct connection with mathematical reasoning and TE ("expérience mentale").[4] Still, these authors, as well as Piaget's own theory of TEs (which he developed in the course of his critical discussion with these authors), have also been unduly neglected so far, both in the general context of today's debate on TEs and specifically with respect to MTEs.

However, it is not my purpose to explain this historiographical neglect, nor it is possible, for reasons of space, to discuss all the authors mentioned. Instead, this paper, starting from Piaget's interpretation and criticism, will focus on Edmond Goblot's account of "logical operations" (the term he used for MTEs) against the background of two questions which are central in today's debate on MTEs: 1) What is their relationship to empirical TEs, and 2) Can we ascribe to them a justificatory role in mathematical proofs, or only a heuristic one?

I will concentrate my attention on Goblot for at least three reasons (here summed up in order of importance): first, Goblot's view hints at answers to the above questions; second, Goblot's works generally precede those of the other two authors, albeit only by a few years; and third, both Lalande and Piaget discussed Goblot's works in much more depth.

Although it does not fall within the scope of this paper to explain today's neglect of Goblot's account of MTEs, it should be noted, at least in passing, that one of its main conclusions will indirectly suggest at least one element of the explanation. As we will see, Piaget's interpretation and criticism of Goblot are both historically and theoretically one-sided and insufficiently appreciative. It follows that, at least in the case of Goblot, Piaget's reading is likely to have played a role – perhaps not a large one, but certainly not an entirely negligible one either – in his neglect.

The paper is organized as follows. In Section 2, I will sketch Piaget's interpretation and criticism: Goblot (as well as Rignano and Chaslin) conflated two different types of TE, which not only correspond to two different stages of cognitive development, but which are also different from an epistemological point of view. Sections 3 and 4 will give a brief reconstruction of Goblot's account of mathematical reasoning from the point of view of the two mentioned open questions in the current discussion on MTEs. Goblot developed concepts relevant for answering these questions. First, in spite of their common constructive nature, empirical and mathematical thinking remain qualitatively distinct. Indeed, contrary to Piaget's charge of confusion between two fundamental kinds of TE, Goblot drew a distinction so similar to that drawn by Piaget himself, that one is strongly led to the conclusion that the first exerted a profound influence upon the latter.[5] Second, Goblot's account of MTEs, at least in a general sense, anticipated the thesis of their visual, iconic, or diagrammatic character. Finally, starting from the constructive, operational and discursive nature of mathematical reasoning he developed ideas about the relationship between psychology and logic in mathematical reasoning that are still worth attention and point toward a justificatory role for MTEs.

2. Piaget's criticism of Goblot's account of MTEs

Already in his "Essai sur la multiplication logique et les débuts de la pensée formelle chez l'enfant", Piaget became interested in Mach's, Goblot's, Rignano's, and Chaslin's account of TEs (Piaget 1922). From a theoretical point of view, Piaget's interest in TEs is hardly surprising: when "formal thought" ("pensée formelle") makes its first appearance (from 11-12 years onward), children begin to develop hypothetical-deductive thinking and are able to infer consequences which are not just derived from memories of past experiences, but necessarily follow from hypotheses (Piaget 1922, 222).

On the one hand, Piaget in several works credited all the authors mentioned above, including Goblot, with having understood thought operationally, that is, from the point of view of the role played by actions and operations in our life: by the notion of "mental experiment" (in French, "expérience mentale"), they assumed that "the essential characteristic of logical thought is that it is operational, i.e., it extends the scope of action by internalizing it." (Piaget 1947/1950, 34). It is no accident, therefore, that, as we will see below, also Piaget uses the term “operation” to synonymously indicate both hypothetic-deductive arguments and MTEs.

On the other hand, however, Piaget raises against Goblot various objections. For our purposes, Piaget’s most important objection is that Goblot conflates two very different types of TE. This distinction, which Piaget sketched in the mentioned article of 1922, essentially corresponds to that 1between two stages of childhood development. The “first stage” (“premier stade”, which Piaget also called “stade irréfléchi”, since observation, as compared with rational reflection, prevails: cf. Piaget 1922, 231 and 238) coincides by and large with “what Claparède has called empirical intelligence, a simple groping with pseudo-choice determined by external circumstances.” (Piaget 1922, 242) In the stage of “formal reasoning” (“raisonnement formel”), also referred to as “stade réfléchi” and “stade de la réflexion” (cf. Piaget 1922, 232 and 242), children, from the age of 11-12 years, conquer the concept of logical necessity, which, according to both Piaget and Goblot, is at stake in MTEs and presupposes hypothetical-deductive thinking, that is, the ability to infer consequences from hypothetical propositions. According to Piaget, the first stage corresponds to TEs in the sense of Mach, Rignano, and Goblot, a sense which is not sufficient to understand the stage in which, starting from 11-12 years, formal thought and the notion of necessity appear (Piaget 1922, 230; see also p. 228).

This distinction will reappear, in a more explicit form, in Piaget 1924/1928 (see e.g. pp. 189-190) and will be developed later, for example in Piaget 1950, discussing the notion of number in children. In the sense of Mach, Rignano, and Goblot, the notion of “TE” (*expérience mentale*) expresses the fact that any action materially executed may be interiorized later in an “imagined experiment”, in which only external variations, even if imagined internally, are involved. On the contrary, in order to understand mathematical necessity, we have to include the actions of subjects who have the ability to hold some facts constant and vary others: “Epistemologically, some thought experiments (I) simply consist in imagining a reality external to the subject […]. Other thought experiments (II), on the contrary, do not amount to simply imagining “the variations of facts”, as Mach says, but the actions themselves of the subject who vary the facts, which is not the same thing.”[6]

Although more often raised against Mach, Rignano and Chaslin, the confusion between two different meanings of TE and the charge of constantly passing “from one of these meanings to the other” (Piaget 1950, 64; see also 63-66, and Piaget 1947/1950, 35 and 92-93), are extended to Goblot in a certain number of passages. The following is one of the most important: “it is in virtue of the laws of the reversible composition that […] the “construction” which originates reasoning is regulated from the start, and it is this internal regulation which determines the choice of the initial propositions. Neglecting the existence of this *reversible* composition of operations, Goblot’s solution turns out to be insufficient to conciliate fruitfulness and rigor, since, then, it results in confounding operations with any material (or mentalized [*mentalisées*]) operations.” (Piaget 1973, 283; emphasis added; see also 1922, 230; 1924/1928, 190; 1926/1929, 384).

As the passage already suggests, and as we will see in detail later on, “reversibility” as a criterion of distinction between two different kinds of TE is one of the pivots around which Piaget’s criticism of Goblot turns. Therefore, it will be worthwhile to spend some time describing it.

Both Goblot and Piaget use the term “operation” synonymously to indicate both hypothetic-deductive arguments and MTEs, even though Piaget gives to “operation” a richer (though not always clearer) meaning. For Piaget, TEs arising out of immediate perception contain irreversible elements. Only “thought experiments (II)”, which Piaget also calls “logical experiments”, reconstruct and transform the “actions” performed by a subject into intersubjectively testable “operations”, which are reversible. “Operation” is to be distinguished from the more general concept of “action”. One of the most elaborated definitions distinguishes four important features of “operations” (see Piaget and Inhelder 1966/1969, 96-97):

1) The internalization of actions: the subject can reconstruct actions mentally instead of actually performing them (Mach’s TE satisfies this one requirement);

2) They are “reversible”, that is, are “capable of a possible return to the point of departure”;

3) They are part of a system of operation, that is, “they are never isolated but are always capable of being coordinated into overall systems”;

4) Finally, the operations of a subject “are not peculiar to a given individual”, they are “common to all individuals on the same mental level”.

These four aspects are closely interwoven with each other. For our purposes it is crucial to consider the relationship between “reversibility”, the first feature of operations listed here, and what might be called the social-intersubjective aspect of “operations”, which is expressed here by the fourth feature (both may also be found in Piaget’s first writings: see e.g. Piaget 1924/1928, 169-180).

Piaget distinguishes two forms of reversibility: inversion (or negation) and reciprocity (see e.g. Piaget & Beth 1961/1966, 176-190; on reversibility, see also Piaget 1946/1970, 262, and 1950/1954, 209-210). These two forms of reversibility are “combined” in the purely formal-mathematical stage (11-12 to 14-15 years), which is characterized by “hypothetico-deductive operations”, that is, by reasoning “about a proposition considered as an hypothesis independently of the truth of its content” (Piaget & Beth 1961/1966, 180).

But the reversibility that characterizes the formal-mathematical stage is only reached through a series of intermediate stages. One of the best-known examples of reversibility as inversion in an intermediate stage is the experiment in which a small ball of clay is transformed into a sausage: children reach an important intermediate stage of their cognitive development when they understand that a perceived change in form can be canceled by an inverse or negative thought operation (the sausage can become a ball again), so that the change in form does not imply that the amount of clay has changed (see e.g. Piaget 1975/1985, 38 and 94-98).

Already at this intermediate stage, and *a fortiori* in the purely mathematical stage, the conquest of reversibility as "inversion" is closely connected with social-intersubjective testability, which we have listed above as the fourth characteristic of Piaget's "operations". In fact, only by conceiving the possibility of returning to the starting point of an action can we confirm a statement about the result of a similar action in the past or discover the possible error contained in it. In other words, reversibility as "inversion" depends on procedures that appear to be intersubjectively invariant on the basis of results that are assumed as reproducible in principle.

The same applies to "reciprocity", the second form of Piaget's reversibility. While inversion corresponds to addition or elimination of elements, reciprocity is the "source of the logic of relations" (Piaget 1932/1932, 79-80). More precisely, reversibility as reciprocity is intimately connected to "objectivity" as "relativity on the physical plane and reciprocity on the social plane" (Piaget 1956/1959, 161). On the one hand, thanks to reversibility, spatial relationships such as left-right and before-behind are relative. On the other hand, reciprocity, as social-intersubjective testability, is the opposite of "egocentrism", and it "can only be developed in and through cooperation" (Piaget 1932/1932, 103).

Both in the case of inversion and in the case of reciprocity, in a sense similar to that in which Wittgenstein argued against the very idea of a "private language", "operations" cannot be proper to a single individual: as Piaget notes, the fourth characteristic of operations enters "into both the individual's private reasoning and his cognitive exchanges" and "involves operations comparable to those which each individual uses for himself" (Piaget and Inhelder 1966/1969, 97). In other words, "inversion" and "reciprocity" are necessary ingredients of the intersubjective testability of experiments or arguments, since they are an essential prerequisite for methodically reconstructing and appropriating in the first person other people's points of view.

Thus, given the intimate relationship between "reversibility" (as inversion as well as reciprocity), on the one side, and, on the other, social-intersubjective testability of Piaget's "operations", we can rephrase Piaget's criticism as follows: missing the role played by reversibility, Goblot's account cannot grasp the intersubjectively testable character of MTEs, which therefore are confounded with TEs in the reductionist sense of Mach2.

In the next section, in order to assess Piaget's interpretation and criticism, we will outline the key tenets of Goblot's account of mathematical demonstration, to which, in his opinion, the most rigorous use of the term "thought experiment" was to be confined. As we will see, a historiographical assessment of Piaget's interpretation and criticism is also a significant step toward appreciating the theoretical relevance of Goblot for today's debate about MTEs. For this reason, in the following section, a brief sketch of some open questions in today's debate on MTEs should precede the outline of Goblot's account of mathematical reasoning.

3. Today's Debate about MTEs and Goblot's Distinction between MTEs and Empirical Reasoning

There are at least two important open questions in today's debate about MTEs. First, starting from Mach and going up to the most recent literature, the majority of authors simply assume without discussion that TEs in the empirical sciences and in mathematics are essentially the same (see e.g. Mach 1906, 197-198, Engl. Transl., 144; Mueller 1969; Brown

2007, 2011, and 2017; Van Bendegem 2003; Starikowa and Giaquinto 2018). However, some authors argue that similarities between empirical and MTEs illuminating as they are risk overshadowing important differences (see Sherry 2006, Buzzoni 2008 and 2011). Therefore, clarifying the relationship between formal-mathematical and empirical-scientific TEs is a desideratum in today's debate.

But perhaps an even more important open question is whether MTEs are only capable of playing heuristic roles, that is, whether they are only useful for discovering mathematical proofs, but once the discovery is made they are completely replaceable at the level of formal 'justification'; or whether they can also play some more justificatory role in mathematics.

For Mach, MTEs "precede the conclusive [*definitiv*] thought construction", and the means we usually employ in them (such as change, vanishing and infinite increase of particular elements) "tell us about new properties and promote insight into their connections" (Mach 1906, respectively 198 and 199; cf. Engl. Transl., 144 and 145). More recently, Starikova and Giaquinto have argued that MTEs are to be distinguished in principle from proofs, since they "neither are, nor serve in place of, mathematical proofs of the conclusions reached" (Starikova and Giaquinto 2018, 276). And in the same vein, Van Bendegem 2003 had maintained that MTEs do not provide proofs, but only lead to an insight into "what a proof could look like", or, if a proof is already available, to a better understanding of it.

For other authors, however, there is no epistemological break, but only a difference in degree between MTEs, which use images or diagrams in a relatively striking way, and formal proofs, which reduce their use to a minimum (in this sense, see above all Brown 2011 and 2017, Sherry 2006, and Buzzoni 2011).

Against the background of these questions, we can move on to a brief discussion of Goblot's theory of MTEs. As we will see, concerning the first question (against Piaget's interpretation), Edmond Goblot can help us to draw a qualitative distinction between TEs in mathematical and in empirical thinking, and, importantly for today's debate, without renouncing the iconic, visual or diagrammatic character of the first. As for the second question, Goblot can help us to ascribe to MTEs not only a heuristic but also a justificatory role, because he pointed out – again in tension with Piaget's interpretation – a plausible relationship of unity and distinction between the logic and the psychology of knowledge. In one sense he defended the irreducibility of logic to psychology, but, in another sense, he rejected their separation: logical order and natural-psychological order cannot be independent, otherwise intersubjective controllability would be undermined.

The general problem of explaining how mathematical reasoning reconciles "fruitfulness" (*fécondité*) and "rigor" (*rigueur*) is at the center both of Goblot's and of Piaget's reflections concerning mathematical reasoning (see e.g. Goblot 1918, IX and XXI; for Piaget, see e.g. Piaget 1950, 59-60, 135, and 276).

Goblot's most concise answer is as follows: "To reason, is to construct" (*Raisonner, c'est construire*) (Goblot 1918, 20). A3 little less concise is the following summary of the fundamental theses of his *Traité de logique*: "TO DEDUCE IS TO CONSTRUCT. Only hypothetical judgments are demonstrable, we demonstrate that one thing is the consequence of another. […] The conclusion is necessary, although it introduces novelty, not because it is *contained in* the hypothesis, but because it is derived by *regulated* operations [opérations *réglées*], that is to say, operations which are not arbitrary. And what rules do these operations follow? Those of formal logic? Certainly not, they are rather propositions already accepted, either in virtue of preceding demonstrations, or as definitions and postulates."[7]

Just from these few remarks we may understand the centrality of the idea of "constructive operations" (*opérations constructives*) in Goblot's work. In a very general sense Mill and Mach's thesis that all our cognitive claims, including those of mathematics, are ultimately related to experience is taken up not only by Rignano (1923, 87) and Chaslin (see e.g. 1926, 212), but also by Goblot (1918, 99). However, against Piaget's interpretation, Goblot's view is in many respects the antipodes of Mill, Mach and Rignano's empiricism. First, unlike traditional empiricism, for Goblot mathematical certainty cannot be explained on the basis of a faithful reconstruction of facts in the mind. The only way to explain it is – in large agreement with Peirce's and Dewey's pragmatism – to go back up to the intersubjective manual actions, constructions, and operations of everyday social life, which are the starting point of even the most abstract reasoning: 4"The fundamental operations of arithmetic are also the logical operations corresponding to the manual operations that the ancients had to perform when calculating by combining and separating stacks of tokens on a table" (Goblot 1918, XX). Secondly, for Goblot no reduction of "logical operations" to "manual operations" is possible. The ideal transposition of real manipulations, operations and constructions creates a hiatus between mathematical objects and reality: "the figure formed by the straight lines joining the centers of gravity of three celestial bodies has no reality outside the mind which conceives it" (Goblot 1918, 130; see also Goblot 1898, 15). However, for a manual operation to become a demonstration, it is not enough, either, merely to become a "mental operation [*opération mentale*], a movement carried out only in imagination"; it must become a "logical operation [*opération logique*]." (Goblot 1918, XX)

The distinction between "manual" and "logical" operations, running parallel to that between empirical and demonstrative-mathematical knowledge, corresponds to Piaget's distinction between the "expériences mentales (I)" and the "expériences mentales (II)". For this reason, it is simply false to say with Piaget that MTEs are, for Goblot, just a reasoning that grows up by trial and error, by experiments which are "simply thought" (*'simplement pensée'*) (Piaget 1922, 230). Instead, we ought to say that Goblot anticipated, and even influenced, Piaget's distinction. According to Goblot, one kind of TE, exactly as in the case of

Piaget, is only "a movement performed in imagination"; only the other kind of TE has already entered the field of logical necessity.

Roger Daval and George Guilbaud (quoted with approval by Piaget in connection with his criticism of Goblot: 1950, 278) also neglected Goblot's distinction between "manual" and "logical" operations. Were this assumption justified, it would be correct to conclude that what Goblot wrote about "operations" is "a strange palette" of statements that are inconsistent with one another (Daval and Guilbaud 1945, 53-54): a conclusion very close to that of Piaget. However, as we have seen, Goblot's distinction between, on the one hand, TEs as "logical operations" (*opérations logiques*), and, on the other hand, TEs as mental reconstruction of the manual operations of everyday life, is generally well-established and consistent.

Moreover, many passages have no coherent meaning if this distinction is not taken into account (see e.g. Goblot 1911, 67; 1918, 263 and 272-274). This holds true at least since the review of Mach's *Erkenntnis und Irrtum*, which Goblot wrote in 1906. Here, Goblot clearly drew this distinction – though not yet using the term "opération logique" – and, in the last analysis, even anticipated the same objection against Mach that Piaget will raise not only against Mach and Rignano, but also against Goblot: "By thought experiment [*expérience mentale*], Mach understands a hypothesis whose verification is obtained by a mental construction […]. By the many examples given by him, he arrives at enlarging the notion to such an extent that he brings in almost all that is called reasoning in general. I have also used this expression of thought experiment, but in a more restricted sense, which I think it is necessary to preserve." (Goblot 1906, 533)

Now, to my knowledge Goblot is wrong to claim that he had previously used the term "expérience mentale": the term he had surely used was "opération logique". But in any case, like Piaget later, he criticizes Mach for using the term "Gedankenexperiment" in a way that conflates two types of reasoning that are different in principle: the empirical, and the demonstrative-mathematical. According to Goblot, it would be better to reserve the term "Gedankenexperiment" (that is, "expérience mentale") for the second type of reasoning, in which demonstrative necessity is obtained: "How can thought experiments provide apodictic certainty? How can they be a *ascertainment [constatation*] *of the necessary*? […] However it may be, I would very much like to reserve the term *thought experiment* for this logical and very important fact, which is perfectly designated by it and to which it seems hardly possible to give a different name." (Goblot 1906, 533)

It is important to note at this point that, in raising this objection, as early as 1906 Goblot not only wants to draw a qualitative distinction between TEs in mathematical and in empirical thinking; he claims that this can be done by a construction from singular (individual, or special) to general cases: "When, […] in order to demonstrate cases of equality of triangles, we superimpose them in thought, we perform an operation which is not a deduction, but an experiment [*expérience*] (in the sense of the English or the German *Experiment*). This experiment does not need to be realized objectively; the objective experiment, performed on individual triangles, would not even have the apodictic value of a thought experiment [*expérience mentale*]. It is thanks to thought experiment that the order of propositions, in mathematics, can pass from the special to the general." (Goblot 1906, 533)

As this passage already suggests, and as we will see in detail now, it is not only Goblot's arguments in favour of distinguishing between empirical experiment and MTEs that could be helpful in today's debate about their relation. There is an equally important contribution in his claim that this distinction cannot be made without singular or individual elements, which are therefore essential to MTEs.

Goblot points out that "logical operations" (which is the term that, as we have seen, he uses for mathematical proofs and MTEs) cannot be understood in terms of deduction as traditionally defined, that is, as based on the passage from the universal to the particular.

Deduction in the traditional sense does indeed explain the necessity of mathematical reasoning, but not its fruitfulness: knowledge of the particular cases would already be implicitly contained in the premises, so that nothing new would be produced or learned (Goblot 1918, 253). According to Goblot, reasoning in a MTE follows the opposite path, though it has nothing to do with induction. It starts from the singular or the special to reach general conclusions (cf. Goblot 1918, 274-275).

This thesis – which bears some significant similarities with Peirce's reflections on diagrammatic reasoning – is a contribution to the theory of MTEs that should not be underestimated. Its importance may be recognized if it is related to the debate of the last decades on the graphic or diagrammatic character of mathematical proofs (see e.g. Brown 2011; Panza 2012; Giardino 2013; Meynell 2018). But also many of Goblot's contemporaries recognized the originality and importance of this thesis. It was taken up by Rignano (1923, 148) and Chaslin (cf. 1926, 6 and 216), who regarded it as a decisive contribution to understanding the specificity of MTEs. But even more independent philosophers, such as André Lalande (1919b, 141-142), Jean Nicod (1919, 382-383), Leonard James Russell (1919, 83), and Piaget himself (1924/1928, 190), realized its importance and originality.

The idea that mathematical proofs or MTEs require an intuitive element as a starting point is closely connected with their constructive and operational nature. If one cannot demonstrate except by operating, mathematical proofs require an intuitive element in the form of "a singular figure", both because operations cannot be performed, even mentally, without such an element, and also because their results cannot be "ascertained" [*constaté*] except on the basis of it. Goblot illustrates this double necessity for "a singular figure" by the construction involved in a TE which demonstrates the equality, in an isosceles triangle, of the angles which are opposite to the equal sides:

> "in order to demonstrate that, in an isosceles triangle, 5the angles which are opposite to the equal sides are equal, we leaf through, so to speak, the triangle, detach it from itself, by thought, and reapply it, turning it over, on the trace we suppose it left on the board. We observe then that the angle between the equal sides necessarily coincides with its own trace, that each side of this angle coincides with the trace of the other side which is equal to it. The coincidence of the third side results from the principle that two points can be joined only by one straight line. Finally, we ascertain [*constate*] that each of the angles that are opposite to the equal sides coincides with the trace of the other." (Goblot 1918, 203)

Although this "ascertainment" [*constatation*] "by thought" distinguishes a "demonstration" from a "manual operation", it requires a reference to a singular reality: "The demonstration consisted […] in an *operation* and in the *ascertainment* [*constatation*] of the result obtained. It goes without saying that this is not a manual operation […] but a mental operation, and that it is not a question of a *physical* ascertainment [*constatation*], such as that one could make with measuring instruments, but a *logical* ascertainment. […] The fact is that one cannot demonstrate except by operating; but an operation (construction, superposition, rotation, etc.) cannot be performed, even mentally, and its result cannot be ascertained [*constaté*] except on the basis of a singular figure." (Goblot 1918, 263-264)

To sum up what I have been saying in this section, Goblot developed some ideas which are still worth attention today. Contrary to the interpretation and criticism by Piaget, he tried to grasp at the same time the continuity and qualitative difference between, on the one hand, empirical "operations" (also of the geometrical or arithmetical kind) that occur in everyday life, and, on the other hand, the corresponding "logical operations", which, though equally starting from singular cases, are able to acquire mathematical-demonstrative character and to which he tended to reserve the term of "thought experiment". On the one hand, Goblot drew a qualitative distinction between empirical and mathematical thinking, and, though with different terminology, anticipated and even influenced Piaget's distinction between two kinds of TE. On the other hand, Goblot's MTEs are intimately connected with a visual or iconic character, which his intensively discussed by several philosophers today.

But what about the other open question I identified at the beginning, concerning the heuristic vs. justificatory role that MTEs can play? As we will see in the following section, Goblot developed ideas about the relationship between psychology and logic in mathematical reasoning that still deserve attention and point toward a justificatory role for MTEs.

4. Logic and Psychology of Knowledge Between Goblot and Piaget

The question whether it is possible to distinguish between the logical validity or truth-value of our discourses and the corresponding empirical-psychological reality of the constructive passages in a human mind is a question that was as urgent for Goblot as for Piaget. The existence of a problem common to Goblot and Piaget is already signaled by the use of the word "génétique". Made famous by Piaget's "genetic epistemology", the word is also central in Goblot, since it refers to everything that involves constructive operations (cf. Goblot 1918, 131). It is no accident that "a broad genetic view" already seemed to Jean Nicod, one of the first reviewers of Goblot's *Traité*, the "most striking feature" of this work (see Nicod 1919, 380).[8]

If we confine ourselves to the years before the so-called "New Theory" (that is, when the quotations from Goblot by Piaget most often appear: cf. fn. 4 of this paper), this is an important point of agreement between our authors, but, as we will see, Goblot provides us with a much more developed and convincing account of the relationship between psychology and logic of knowledge. In the preface to the second edition of the *Introduction à* *l'épistémologie génétique* (vol. 1, devoted to mathematics), Piaget explicitly considers the objection that the *validity* of knowledge is one thing, connected with normative considerations, while the *genetic* process is another, because it consists only of strictly factual conditions, and, therefore, can have no truth-value in itself (cf. Piaget 1950, 8).

Kant's (and Frege's) well-known distinction between *quid facti* and *quid juris*, genesis and validity (or truth), psychology and logic of knowledge – taken up for example by Popper and the logical empiricists (see e.g. Popper 1935, 31 and Reichenbach 1951, 231) – is a formidable challenge to the very idea of a "genetic epistemology", according to which epistemological concepts such as those of number, causality, permanence of objects, spatio-temporal localization, etc., cannot be understood apart from their genetic-developmental reconstruction.

Against the cogency of the distinction, Piaget argues that the object of genetic epistemology is not "any psychological succession of simple states of consciousness", that is, a chain of simple ideas linked together, but a self-directed process of connecting ideas by the knowing and acting subject, which is the supreme epistemic authority and the ultimate source of norms (Piaget 1950, 8-9).

Goblot's position is similar, but much more convincing. On the one hand, for him it is impossible to reduce logical operations to mental reconstructions of real processes, since logic is not the science of the "operations of the mind" [*opérations de l'esprit*], but rather the science of the operations of the mind "as they lead to the truth" (Goblot 1918, 13). Logic cannot be reduced to the results of the deductive mental passages, since it rules them, providing the standards to evaluate them: "the logical relation is distinct from the process that it dominates and rules" (Goblot 1914, 342). In this sense, Goblot preserves the irreducibility of the logical to the psychological, which Rignano's radical empiricism tended to dissolve.

On the other hand, for Goblot logic and psychology are not different in the sense that the psychologist investigates “under what conditions there is belief”, while the logician investigates “under what conditions it is justified.” For, “if logical order and natural order are thus independent, logic gets no hold on thought; intellectual processes offer an impenetrable automatism to reason; they are all they can be. Reason is foreign to intelligence” (Goblot 1918, 14). For Goblot, developing a psychology of mental operations means above all denying that mathematical truths and realities lie outside the human mind, out of a connection with mental operations performed in the first person. He refuses, in other words, the real existence of logical processes that, because of their reality, would be independent of the subject in which they occur. I would say that, if compared with Piaget, Goblot stood on a much more personalist line: “To put logic outside of psychology would be to put reason outside of intelligence, truth outside of knowledge. [...] the conditions of truth are other than the conditions of conviction. However, the reasons are only ideas capable of convincing, that is, of forcing the admission of other ideas, and this force of the proof cannot be conceived outside of a mind in which it resides and on which it acts, since the proof, the proven assertion and the determination of the proven assertion by the proof are operations of the intelligence.” (Goblot 1914, 343-344; almost identical passage in Goblot 1918, 20-21)

Lalande's charge of “psychologism” (1919b, 132) is based on a complete misreading of this passage (and few other statements) taken out of context. Goblot’s claim that “[t]o put logic outside of psychology would be to put reason outside of intelligence”, did not intend to reduce logic to psychology, but rather to defend, against Platonism, a procedural notion of reason. It is no accident that this misunderstanding is also to be found in Piaget (1949, 13). Lalande was one of Piaget’s teachers during his studies in Paris (see Piaget 1952, years 1918-1921), and there is a close connection between this misunderstanding and Piaget’s charge that Goblot would have reduced MTEs to empirical-psychological processes in Mach’s sense.

According to Goblot, the only way in which the logical validity may be defended as autonomous is through a constructive procedure which is meaningful, and strictly speaking exists, only insofar as it is actually performed by personal subjects that, on the basis of assumptions or rules they have decided to accept, reconstruct and appropriate in the first person (in this sense, "psychologically") the steps that lead to a certain conclusion: "We know no thought that is not discursive. An intelligence that –not being subject to the law of time, as our intelligence is –would immediately perceive the consequences contained in the principles and which would be purely intuitive, such an intelligence is not only an ideal which is inaccessible to us; it is not only something 7absolutely unknown to us; it is something absolutely impossible, because it is contrary to the essential nature of reasoning." (Goblot 1914, 343; see also Goblot 1918, 19)

From this point of view, the distinction between logic and psychology of knowledge must be asserted in one sense and denied in another: "One cannot perceive the consequences in the principles, because they are not contained in them. To reason is to construct. The distinction between the logical and the psychological disappears and reappears to disappear again. It always reappears because it is in part justified; since it is nothing but the distinction between reason and reasoning. It disappears and seems to be empty and illusory as soon as one attempts to separate reason from reasoning, or reasoning from reason, and to make them the subject matter of two different sciences." (Goblot 1914, 343; cf. 1918, 19)

As I shall now briefly argue, this thesis does not only confirm the one-sidedness of Piaget's interpretation and criticism of Goblot's account, it also represents another valuable contribution to today's debate about MTEs.

As we saw in Section 2, Piaget's charge that Goblot confused two types of TEs is based, above all, on the supposed lack of the notion of reversibility, which is closely interwoven with that of social-intersubjective controllability. But it is exactly this that Goblot defends when he challenges the concept of a mathematical reasoning in itself, apart from a psychological-personal and temporal construction. As Goblot writes in a passage (of which we have already quoted the first sentence):

> "the logical relation is distinct from the process that it dominates and rules; the relation of the principle to the consequence is independent of the act by which the mind grasps it. This relation is not a process; it does not happen in time. [...]. However, "logical anteriority" is not at all anteriority. In an isosceles triangle, the equality of sides is not anterior to the equality of angles. What explains this metaphor is that our discursive thought is obliged to admit the consequence *after* having admitted the principle: the timeless order of logical dependence prescribes to the thought the temporal order of its discursive assertions." (Goblot 1914, 342-343; cf. the same passage in 1918, 19)

On the one hand, the process, whereby we develop not only formal proofs, but also reasoning in general, is something which takes place in time and in the individual mind of a person who thinks: it is made up of a series of steps which follow each other. On the other hand, the relationships from one step to another are not subject to the unidirectionality of the time and therefore are in principle reversible in the sense that they may be reconstructed from the start by anyone who cares to check them.

This interpretation is strongly supported by another of the more original points of Goblot's philosophy, which I will call the functional equivalence between the universality of mathematical (or scientific) statements and the doctrine of imperatives. Goblot simply reverses the view expressed by Hume's law: "nothing is simpler than to transform into a rule the very statement of a theorem or a natural law. Every theorem, every law is the expression of a necessary relation; as soon as the first term is, directly or indirectly, in the power of man, it is a means of realizing the second: *Theorem*: The product of a sum by a number is equal to the sum of each term by this number. *Rule*: To multiply a sum by a number, multiply each term by this number and sum the resulting products." (Goblot 1918, 3-4)

It follows from this that the truth of a general proposition can always be translated into 'technical' rules, the necessity of which coincides with their reproducibility in principle, which is assumed to be identical by every possible knowing subject. Any operation controlled only by rules or by "propositions previously admitted," can be repeated as often as one likes, and always with the same result:

> "general propositions, which are truths only when we confine ourselves to contemplating them, become *rules* when we operate [...]; an acting and operating thought that takes these truths as practical rules of its action, can ascertain [*constater*] the new result which it has itself constructed. [...] The ascertained result [*résultat constaté*] is necessary to the extent that it is determined by the application of rules. It remains contingent and modifiable to the extent that it depends on the singularities of the example chosen. And that is why it is general. The geometer always has in mind, when reasoning on a figure, the distinction between the properties of the figure which are formally stated in the hypothesis and the properties which, not being specified, remain indefinitely variable. Any operation which is regulated only by the first may be repeated, with the same result, on any different figure which realizes the hypothesis, whatever its singular properties may be." (Goblot 1918, 264-265; see also pp. 165-166)

What Goblot says here, with reference to the particular case of "logical operations" (that is, in his language, MTEs), clarifies one of the fundamental aspects of Piaget's "operations", that is, their intersubjective testability or reproducibility, and, at the same time, brings to the fore a sense in which Reichenbach's and Popper's separation of discovery and justification must be rejected. The logical positivists and Popper failed to realize the sense in which a certain 'genetic' attitude is essential for justification. If we want to test the truth of a statement, we must adopt a *genetic-reconstructive* attitude and retrace the operations performed and communicated by those who first obtained a certain result by means of those operations (for a more detailed treatment of this point, see Buzzoni 2008, above all Ch. 1, Sections 4-6, and Ch. 3, Sections 4-6).

Now, in so far as one of the essential ingredients of Goblot's view about MTE is the reproducibility in principle, at any time and by any concrete (psychological) person, of the procedural steps leading to its conclusions, we find in him not only an anticipation of Piaget's concept of mathematical "operations", but also the [8]invitation to abandon the distinction, at least if it is understood as a distinction in principle rather than in degree, between a heuristic and a demonstrative use of reason. In my opinion, Goblot's rejection of the separation of psychology and logic implies that MTEs cannot have [9]an exclusively heuristic value, since they cannot intentionally formulate a hypothesis without an implicit or explicit connection to reasons that can [10]be checked in the first person. This applies to any human discourse that is not the product of a chance process, but applies a fortiori to MTEs, which must already contain some justification, implicit or explicit, of the theoretical hypotheses that they formulate; and if the justification turns out to be insufficient, this will necessitate the search for other, more convincing reasons in favour of them. In other words, with respect to the principle of the intersubjective reconstructibility of the methodical steps of which they are constituted, MTEs should not be qualitatively distinguished from formal demonstrations[11], as traditionally understood. For this reason, we should recognize that MTEs may play a role of justification, however little it might in particular cases be.

That, according to Goblot, all mathematical proofs (or TEs in the strict sense) require an intuitive-singular element as a starting point, is more testimony in this direction. The need for an intuitive-singular element does not (and cannot) support the claim that MTEs have 12an exclusively heuristic value, because Goblot effectively reconciles it with mathematical necessity. On the contrary, this supports the opposite claim, because the intuitive-singular element counts as an essential part of mathematical reasoning. It is thus no accident that Goblot reiterated exactly this point when discussing the connection between psychology and logic. The visual or graphic element plays a necessary role, both "genetically" and logically, in explaining the generality and certainty of mathematical proofs: "In geometry, the importance of graphic constructions has escaped no one, but logicians are inclined to see them as mere auxiliary or preparatory operations to reasoning. But they are the reasoning itself. The same applies to all pen and ink operations [*opérations de plume*] which intervene in the algebraic reasonings." (Goblot 1918, 273; Chaslin (1926, 216) echoes this passage almost *verbatim*)

Affecting the letter but not the spirit of Goblot's account (in which there is no distinction of this kind, perhaps only because Goblot wished to insist on the unity of mathematical reasoning), it might be advisable to regard MTEs as a particular type of argument within the wide family of possible forms of mathematical thinking: certainly not as a qualitative difference, but only as one of degree: between, on the one hand, mathematical proofs making explicit use of diagrams or images (which we might designate by MTEs) and, on the other hand, more abstract mathematical (or metamathematical) proofs, which reduce their use to a minimum (sometimes to the point of giving us the illusion that they have been completely eliminated).

From this point of view, Goblot's considerations can be interpreted somewhat in the sense of James Robert Brown's concept of proof. Brown claims that "proof = explanation +

guarantee", provocatively using the word "guarantee" in the sense in which a new toaster comes with a guarantee: "not a promise that it will work perfectly. Rather, […] the promise that it will work or it will be fixed or replaced or our money will be refunded." (Brown 2017, 48) Although Goblot's view is very different from Brown's both in rejecting mathematical Platonism and in distinguishing in principle the formal from the empirical use of reason, his rejection of the distinction between psychology and justification is an important contribution to the spirit of Brown's position, according to which, within mathematical research, there is room for statements that can be guaranteed in a weaker and much more differentiated sense than that of formal derivation. That, however, is a story for another paper.

5. Conclusion

Let us sum up the main results of this paper gathering its two threads, historical and theoretical, together. Edmond Goblot, Eugenio Rignano and Philippe Chaslin have been unduly neglected in today's debate about MTEs. To partially redress this neglect, I have historically and analytically reconstructed some aspects of Edmond Goblot's account of "logical operations" (the term he used for MTEs) and its interpretation by Piaget against the theoretical background of two questions which are central for today's debate on MTEs: 1) what is the relationship between mathematical and empirical TEs, and 2) do MTEs play a merely heuristic, or also a justificatory role in mathematical reasoning?

With regard to the first question, Goblot's arguments in favour of a qualitative distinction between TEs in mathematical and in empirical thinking are still worth attention, not only because of their intrinsic strength, but also because, at least in a general but important sense, they anticipated the thesis of the graphic, visual or iconic character of MTEs, intensively discussed today. Goblot combined the distinction in principle between empirical and mathematical reasoning with the thesis that mathematical proofs always require an intuitive and singular element as their starting point, from which, by construction, a general conclusion can be reached (Section 3).

With regard to the second question, Goblot's account of the relationships between psychology and logic suggests that it is not possible to ascribe to MTEs only a pre-scientific, heuristic validity. In one sense he defended the irreducibility of logic to psychology, but, in another sense, he rejected the separation between psychological construction and mathematical justification: in this last sense, to ascribe only a psychological-heuristic role to MTEs would undermine the most important condition of knowledge, that is, intersubjective testability (Section 4).

The historical-theoretical corollary is that Piaget's interpretation and criticism are fundamentally flawed. First, contrary to what Piaget affirms (and contrary to his inclination to overlook the influence of Goblot on the development of his thought), Goblot clearly anticipated, apart from terminological preferences, Piaget's distinction between different kinds of TE: empirical vs. mathematical. Second, again in tension with Piaget interpretation and criticism, Goblot not only anticipated, but better justified, one of the most important features of Piaget's MTEs. For Goblot, the operational-constructive nature of MTEs reconciles, to the best of our powers, the necessity of a procedural and psychological aspect of mathematical proofs with the social-intersubjective use of reason.

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1 Eugenio Rignano (1870-1930), professor of philosophy at the University of Pavia, was editor of the periodical *Scientia*. Rignano's major work (1920/1923) was read and appreciated internationally. Rignano's work in the philosophy of biology and economic policies attracted international attention as well (Ball 1904; Harvey 1909).

2 Edmond Goblot (1858-1935), professor of philosophy at the University of Lyon, was a French logician and philosopher of science. In his major work, Goblot (1918) developed ideas expressed in earlier artides (1911, 1914). As he informs us ("Avertissement de l'auteur"), this work was written, with the exception of the last chapter, by July 1914. Goblot (2010) later devoted himself to sociology, and this book remains an important point of reference (Lallement 2015). Ducret (1984, 2:642-63) provides a general introduction to Goblot.

3 3. Philippe Chaslin (1857-1923), one of the great French alienists, made a fundamental contribu¬tion to psychopathology by adopting a dassification based on clinical types rather than on diseases or nosological entities (see Chaslin 1912). Later, Chaslin (1926) fully explored the relationship between TEs and mathematical knowledge.

4 Piaget mentions Mach, Goblot, Rignano, or Chaslin in connection with the notion of TE in many works, most of them between 1922 and the early sixties (see esp. Piaget 1922, e.g., 230; 1924/1928, e.g., 181-185, 235-240, 251-252; 1973, 62-67 (on Goblot, see chap. 3, sec. 3); but see 1923/1959, 45, 125; 1926/1929, 384; 1946/1970, 1947/2005, 34; Piaget and Beth 1961/1966, 18-21 [for Goblot]). We may overlook the many changes that Piaget's thought has undergone, the most important of which is perhaps the so-called New Theory (for some recent reflections on this point, see, e.g., Burman 2016), because Piaget's objection discussed in this article—that Goblot would have confused MTEs with TEs in the

sense of Mach—remained essentially unchanged in the works just mentioned. Page references are given to the English editions only, unless I have made modifications to the English translation, where page references are also given to the original French edition.

5 This conjecture is all the more plausible given Piaget's inclination to overlook the influence of Goblot on himself. In his autobiography, e.g., when he speaks of his studies of logic and philosophy of science in Paris (starting from 1919; cf. Ducret 1984, 2:500), Piaget (see 1952, years 1918-21) does not mention Goblot at all.

6 For our purposes, we can confine ourselves to this fundamental distinction. In fact, according to the letter of Piaget's remarks, one should distinguish three kinds of TE since, in the text we just quoted, he divides the TEs (II) in two further subspecies, TEs (II A) and TEs (II B) (see 1973, 66; for a similar distinction, see also 1924/1928, 235). We may abstract from this more fine-grained distinction to the extent that—as we will see below—both kinds of TEs (II) have the common feature of involving re¬versibility, which distinguishes them from Mach's sense of TEs.

7 Goblot 1922, 50-51. The passage, translated into English by W. Mays, is also quoted in Piaget and Beth 1961/1966, 19; Mays's translation has been slightly modified, original italics and small caps have been restored.

8 In general, secondary literature discussed the relationship between Piaget and Goblot only rarely. To my knowledge, there is only one important exception: Ducret 1984 (see e.g. 642-663, 694-695, 758-763, 775-782). Besides reminding us of the fact that, in Lalande's course in Paris, Piaget wrote a report on Goblot's *Traité de Logique* (of which however we do not know the content: Ducret 1984, 649), Ducret rightly points out the role played by the "operational and constructivist" character of Goblot's account "in structuring Piaget's psychological thinking" (cf. e.g. Ducret 1984, 651). He even mentions "reversibility" as a possible influence of Goblot on

Piaget (Ducret 1984, 652-653, 758-759). My considerations are, at least in significant measure, a corroboration and an extension, though only from the limited perspective concerning MTEs, of this more general point made by Ducret. However, there are two points of divergence. First, Ducret (in a fn. on p. 760) recognizes that Piaget is inclined to confound a bit too quickly Rignano's and Goblot's account, but he indirectly reaffirms Piaget's criticism of Goblot, when he says, following Gaston Milhaud, that "the negative side" of Goblot's account is to have identified "operations" (understood as "external actions") and "deduction" (Ducret 1984, 652 and 659-660). 6Second, although mentioning "reversibility" as a possible line of influence from Goblot to Piaget, Ducret says that Goblot's considerations remain "marginal", whereas Piaget not only accords to reversibility "a central place" in his theory (Ducret 1984, 653 and fn.), but also uses the notion "in an entirely original way" (Ducret 1984, 759). In my opinion, as we will see more fully in the course of the present section, the main influence on Piaget concerning the notion of reversibility is to be found in the central theses of Goblot's account of MTEs, only secondarily in the choice of the same word.