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CICLO XXXI

THE EMERGENCE OF NONLINEAR DYNAMICS IN THREE DIFFERENT ECONOMIC MODELS

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To my parents, who inspired me with a lifetime of sacrifice. To Nicla, who supports and tolerates me in the chaos of everyday life. To my uncle Andrea, who transmitted to me his passion for science and irony.

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Chapter 1

Introduction

The main focus of this thesis, as the title suggests, regards *nonlinearities* in dynamic systems which describe economic phenomena. The term *nonlinearity* evokes the appearance of complex dynamics, such as multiple equilibria, cycles, path dependence, global and local bifurcations. In economics, nonlinear dynamics represents an interesting issue in order to provide a theoretical framework for the analysis of phenomena such as economic cycles, financial bubbles, population dynamics, etc. Since the works of Goodwin (1947), Hicks (1950) and Day (1982), nonlinear dynamics emerged within economic modelling. One of the breakthroughs in this research area has been to understand if nonlinear dynamics may occur when explicit behaviours of the agents are considered, defining the decision making either through optimisation or decisional *heuristics*.

The aim of this thesis is to show how the emergence of nonlinear phenomena may take place in different economic frameworks. To this purpose, we will propose different models, related to different themes of economic and social interest, which are common in being formalised as dynamic systems. This tool allows to model the expectations formation of the economic agents involved, that is to describe their expectations on the consequences of their actions and the future state of the economy in which they operate. In addition, the approach of dynamic systems allows considering agents that do not homogeneously behave. Indeed, we will see how such heterogeneity may be associated to different decision-making mechanisms or to heterogeneous *beliefs*, which may switch as a result of changes in the state of the system that the (heterogeneous) agents consider as relevant. Finally, dynamic systems make it possible to analyse the different scenarios that may appear in the evolution of the system as (i) the initial conditions associated to the system change and (ii) the parameters, exogenous or endogenous, that govern the heuristics associated with the agents vary.[1]

A first theme that we want to address in this work is the coevolution between economic activity and the environment. The recent literature on eco-

[1] See Cressman and Ansell (2003); Xepapadeas (2005); Bischi et al. (2009a) for complete surveys on the use of the dynamic systems approach on several economic frameworks.

nomic growth and the developments on the implementation of environmental policies have made studies on short and long term trends in the coevolution between economic and environmental variables a central theme of crucial interest. In particular, the literature, through overlapping generations models, has highlighted the existence of different interpretations. On the one hand, the paradigmatic work of John and Pecchenino (1994) described the economic agents as subjects able to internalise the environmental problem (seen only as a consumption good) and therefore devolve a share of their resources for defensive environmental expenditures; on the other hand, the work of Antoci et al. (2009) has assumed the existence of agents who do not internalise the environmental problem (exclusively taken into account by the consumers) and where therefore the environmental dynamics are exogenously determined. A third approach (see Bovenberg and Heijdra, 1998) has been to assume the government intervention, which considers the environment as a public good, and then finances the environmental maintenance by using a share of the general taxation revenue. In order to bring together the different strands of the literature concerning the dynamic coevolution between economic and environmental systems, and to introduce the assumption that the environment can be seen also as a productive input, the second chapter of this work has a twofold aim: (i) to review the crucial dynamic models of the economic theory and some relevant discrete time dynamic models; (ii) to analyse a discrete time overlapping generations model in which economic activity depends on the exploitation of a free-access natural resource and, in addition, public expenditures for environmental maintenance are assumed, as economic agents are assumed to be unable to internalise the problem of environmental maintenance. By characterising some properties of the map and performing numerical simulations, we investigate long run consequences of the interplay between environmental public expenditures and the private sector. In particular, we identify different scenarios in which multiple equilibria, as well as complex dynamics, may arise and conditions under which an unbounded growth path for both economic activity and the environmental resource may exist.

A second theme we deepen in this thesis concerns the emergence of complex dynamics in duopolistic markets where (heterogeneous) firms interact. By recalling the seminal work of Cournot (1838), some paradigmatic contributions have shown how, depending on the assumption made on demand functions, firms decisional mechanisms and substitutability between goods, complex behaviours may appear in the dynamics of a duopoly market. In this regard, Rand (1978), Dana and Montrucchio (1986) and Puu (1991) have been the first in showing how complex dynamics may emerge in duopolies where some of the classical assumptions are replaced. Afterwards, the focus has shifted to investigate duopoly models in which assumptions on the firms' knowledge of the market demand are relaxed. Indeed, models in which decisional mechanisms which are based on a reduced degree of rationality have been developed and analysed. In particular, the literature has focused on models in which firms adjust their decisions through a gradient-rule (see Bischi et al., 2007) or, as shown in Bischi et al. (2007), adopt the so called Local Monopolistic Approximation. By adopting

such decisional mechanisms, nonlinearities in duopoly dynamics deriving from the interaction of homogeneous firms (i.e., adopting the same decision-making mechanism) have been analysed. The more recent works have then begun to consider the existence of heterogeneity in the duopoly market. Indeed, several works have focused on heterogeneity in the decisional mechanisms of firms (see Cavalli et al., 2015); different works have instead analysed the effects of heterogeneity in the products placed on the market by the two firms (see Agliari et al., 2016). The aim of the third chapter is then to propose a discrete time duopoly model in which firms (i) are heterogeneous in decision-making mechanisms and (ii) produce heterogeneous goods (i.e. we assume a degree of differentiation between goods). In particular, we assume that a firm adopts the local monopolistic approximation (LMA) approach, while the rival adjusts its output level according to the gradient rule (see Bischi et al., 1999). By analysing the resulting two-dimensional map, we derive conditions for the stability of the Nash equilibrium and investigate some bifurcations scenarios as parameters vary. Moreover, we show that different from Agliari et al. (2016), both a high and a low level in goods differentiation may have a destabilising role in the system.
A third theme on which this thesis wants to focus on is the role of women in society and, in particular, how their role in the family and in the work activity may evolve. Indeed, the extent of female contribution to market activities (e.g. work) and non-market activities (e.g. family) continue to occupy a central role in public opinion and in decades-long debates among social scientists. In order to investigate this complex and really deep social issue, we have decided to start approaching it by investigating how women decide to allocate their time between family and work. In particular, we want to analyse how the interaction with other women in the workplace and the perception that society has about their short run decisions (the so called *social norms*) may affect the long run behaviour of such women. By adopting the lenses of the Preference Theory (see Hakim, 2000) and following the insights of Fernández et al. (2004); Fogli and Veldkamp (2011) and Fernández (2013), in which the effects of interaction and social norms in affecting women's decisions are discussed, the aim of the fourth chapter is to propose an evolutionary model (see Bischi et al., 2009b) in which women with different and *adaptive* inclinations may coexist and socially interact. More specifically, we assume a population composed by family-oriented and career-oriented women. The preferences of both types of women are assumed to be affected by extrinsic benefits (e.g. a Pay-for-Performance contract at work), intrinsic costs (i.e. their innate inclination toward spending time at work or with the family) and by social norms. According to *word of mouth* dynamics (Dawid, 1999), women socially interact and compare their different positions, learning about possible payoff differentials. Social interaction, therefore, sparks the evolution of the distribution women types in the population. The analysis allowed us to show that (i) both scenarios in which the two (different) types of women coexist and scenarios in which one of the two types tends to disappear from the population are achievable; and (ii) due to the destabilising role of the intensity of choice parameter (assumed to be a constant), periodic cycles, as well as, chaotic regime may appear.

Chapter 2

Nonlinearities in dynamic coevolution of economic and environmental systems

2.1 Introduction

The literature on economic growth has so far focused on highly stylized models.[1] More recently, an increasing attention has been paid to the analysis of models which consider more realistic interplays among variables and the possible emergence of non-trivial dynamics. In this regard, one of the branches of the recent literature on economic growth has then focused on exploring particular connections between economic activity and the environment. In particular, several models, able to highlight both the long-term trends of the variables as well as the emergence of short or medium-term phenomena, have been proposed.

The aim of this chapter is twofold. First, we review the crucial dynamic models of the economic theory, such as the Solow model, the Ramsey-Cass-Koopmans model, and the overlapping generations (OLG hereafter) structure in which environmental variables are introduced and through which we show how different approaches and assumptions, may affect both the resultant short and long run dynamics. Indeed, the choice of considering the environment as productive input or as a consumption good, the different hypotheses on the agents' rationality (and then the different allocative problems) may have relevant consequences on the models' dynamics. For example, depending on alternative assumptions, the productivity in the private sector as well as the complementarity or substitutability between environmental and private good could be effective or not in defining the complexity of the system. Nevertheless, cyclical dynamics or multistability seem to be likely results whenever models take into account the interaction between economic and environmental systems. Second, in order to

[1]See Barro and Sala-i Martin (2003) for a survey.

bring together different strands of this literature and following the suggestions resulting from several empirical studies[2], we propose a discrete time dynamic model in which, in an OLG framework, we discuss the dynamic effects of the interaction between the economic activity and a free-access natural resource when the latter is assumed to be a production input.
The remainder of the chapter is organised as follows: In Section 2, we analyse two seminal models on this issue. In Section 3, we review several relevant models in a discrete time framework. In Section 4, we present and discuss the local and global dynamics of an OLG model in which the environmental resource is considered as a productive factor. Finally, Section 5 will conclude.

2.2 Two general frameworks

In this section, we review two milestone models in the literature on interactions between economic and environmental systems.

2.2.1 The Green Solow model

One of the first models which highlights the possible existence of nonmonotonic relationships[3] between economic activity and the environment has been proposed by Brock and Taylor (2010).[4] Specifically, the authors consider an extended version of the Solow model in which savings[5] and environmental investment choices are fixed.
In the model, the output Y is generated by a strictly concave production function F with constant returns to scale and where the inputs are the effective labour BL, where B is the labour-augmenting technological progress, and the capital K. Capital accumulates via savings, sY, with $s \in (0,1)$ and depreciates at $\delta > 0$. The rate of labour-augmenting technological progress, $g_B > 0$, determines the growth rate of labour efficiency:

$$Y = F(K, BL), \dot{K} = sY - \delta\, K, \dot{L} = nL, \dot{B} = g_B B. \tag{2.1}$$

The impact of pollution has been modelled assuming that each unit of economic activity, F, generates Ω units of pollution. Then, the amount of pollution differs from the amount generated by the economic activity only when abatement occurs. Moreover, abatement is considered as a constant returns to scale activity and the pollution abated is assumed as an increasing and strictly concave function of both the total economic activity F and the economy's ecological effort, F^A. Then, pollution is defined as

[2] See, for example, Kozluk and Zipperer (2015).
[3] See Aghion and Howitt (1999) for a complete review.
[4] This work has been circulated as a working paper for many years and it was a sort of milestone for the literature of the 2000s on the relationship between growth and the environment.
[5] This assumption is commonly used to simplify the analysis.

$$P = \Omega\, F - \Omega\, A(F, F^A) \tag{2.2}$$

where A is the abatement level. The specification can be rewritten also as

$$P = \Omega\, F a(\theta) \tag{2.3}$$

where $a(\theta) \equiv \left[1 - A\left(1, \frac{F^A}{F}\right)\right]$ and $\theta = \frac{F^A}{F}$. Combining environmental assumptions with the Solow model, the output Y becomes $\widetilde{Y} = (1-\theta)F$ and, assuming also an exogenous technological process in abatement at the rate $g_A > 0$, it is derived

$$\widetilde{y} = f(k)[1-\theta] \tag{2.4}$$

$$\dot{k} = sf(k)[1-\theta] - [\delta + n + g_B]k \tag{2.5}$$

$$p = f(k)\Omega\, a(\theta) \tag{2.6}$$

where $k = \frac{K}{BL}$, $\widetilde{y} = \frac{\widetilde{Y}}{BL}$, $p = \frac{P}{BL}$ and $f(k) = F(k, 1)$.
From a dynamic point of view, the model presents a differential equation in k from which, by assuming that Inada conditions[6] hold for F and θ fixed, the unique interior fixed point k^* attracts every initial condition $k(0) > 0$.
Moreover, it can be noticed that, as $k \to k^*$, the aggregate output, consumption and capital approach the same growth rate $g_B + n$ and, consequently, their per capita levels grow at the rate $g_B > 0$.
At the steady state level k^*, the growth rate of emissions g_P is given by

$$g_P = g_B + n - g_A. \tag{2.7}$$

In order to describe the dynamic relationship between capital accumulation and emissions level, the authors introduce a definition of sustainable growth in these terms: a balanced growth path[7] is sustainable if it is associated to both rising consumption per capita and an improving environment. In mathematical terms, it is characterised by the following conditions

$$g_B > 0, g_A > g_B + n. \tag{2.8}$$

By differentiating the equation (2.6) and assuming $f(k) = k^\alpha$, the dynamics of the model are described by the following system:

$$\begin{cases} \frac{\dot{k}}{k} = sk^{\alpha-1}(1-\theta) - (\delta + n + g_B) \\ \frac{\dot{P}}{P} = g_B + n + \alpha\frac{\dot{k}}{k}. \end{cases} \tag{2.9}$$

Therefore, the system (2.9) allows to capture the evolution of emissions along time, as $k \to k^*$. By assuming sustainable growth (that is, $g_P < 0$), the value k_T

[6]In economic literature, the Inada conditions are assumptions introduced in order to guarantee the stability of an interior equilibrium (see Barro and Sala-i Martin (2003)).

[7]In dynamic modelling, a balanced growth path is a trajectory such that all variables grow at constant (but potentially different) rates.

(called turning point), ensuring $\frac{\dot{P}}{P} = 0$, is lower than k^* and this implies that the time profile of emissions depends on the position of the initial value k_0 related to the turning point. In particular, if the economy starts with an initial capital stock such that $k_0 < k_T$, emissions first rise and then fall. Hence, an inverted U-shaped profile, recalling the environmental Kuznets curve, is obtained. Instead, if an initial capital stock $k_0 > k_T$ is assumed, emissions monotonically fall as k moves towards its steady state value k^*.
Finally, when the sustainable growth is not assumed, emissions grow for all t even as k approaches the steady state value.

2.2.2 The Ramsey-Cass-Koopmans model with environmental pollution

The model in Xepapadeas (2005) represents a natural evolution of the approach proposed in the previous subsection. In this context, consumption-investment choices are derived in a decentralized framework composed by intertemporal utility maximising agents and perfectly competitive profit maximising firms. The individual utility function depends on the per capita consumption flow $c(t)$ and the pollution stock $P(t)$.
The representative consumer considers the pollution level as fixed and solves the problem:

$$\begin{aligned} &\max_{c(t)} && \int_0^\infty e^{-\rho\, t} U(c(t), P(t)) dt \\ &\text{subject to} && \int_0^\infty e^{-R(t)} c(t) dt = k(0) + \int_0^\infty e^{-R(t)} w(t) dt \end{aligned}$$

where ρ is the discount rate in the utility function, $k(0)$ is the initial capital and $R(t) = \int_{\tau=0}^{t} r(\tau) d\tau$, with $r(\tau)$ real interest rate at time τ and $e^{-R(t)}$ an appropriate discount factor. Under standard assumptions of concavity on U and imposing that $\lim_{c \to +\infty} U_c(c, P) = 0$, the solution of the optimisation problem, obtained with the use of the maximum principle, is interior (that is, $c(t) > 0$ for every t) and the consumption path is defined by

$$\frac{\dot{c}}{c} = \frac{1}{\eta}\left[r - \rho + \frac{U_{cP}}{U_c} \dot{P} \right] \tag{2.10}$$

where $\eta = -\frac{U_{cc}}{U_c} c$.[8] By assuming that (i) the production function $f(k)$ satisfies the Inada conditions, (ii) markets are competitive and firms are profit maximizers (which imply $f^{'}(k) = r + \delta$), the economic dynamics is described by the following system:

[8]In order to simplify the notation, the subscripts denote partial derivatives. Therefore, the general equality $V_x(x, y) = \frac{\partial V(x,y)}{\partial x} = V_x$ henceforth holds.

$$
\begin{cases}
\frac{\dot{c}}{c} = \frac{1}{\eta}\left[f^{'}(k) - \rho - \delta + \frac{U_{cP}}{U_c}\dot{P}\right] \\
\dot{k} = f(k) - c - \delta\, k \\
\dot{P} = \phi\, f(k) - mP.
\end{cases} \tag{2.11}
$$

By analysing the model, a unique steady state (c^*, k^*, P^*) exists and the dynamics, generated by an optimisation process, evolve on the stable manifold of (c^*, k^*, P^*) and converge to it. As explained in Xepapadeas (2005), by considering that pollution evolves, it can be noted that the coordinates (c^*, k^*) are generally not affected by the stationary state value of P^*, although the approach path to the steady state is affected (due to the presence of $\frac{U_{cP}}{U_c}\dot{P}$ in the first equation of system (2.11)).

2.3 Discrete time models

If, on the one hand, the aforementioned models are suitable to describe long run trends, on the other hand they are not able to capture the occurrence of short or medium run nonlinear phenomena.
Regarding this point, the work of Day (1982) provides an example of how complex phenomena may emerge from the dynamic analysis of a simple economic framework.

2.3.1 The Day's (1982) model

In this work, the author considers a neoclassical growth model à la Solow (1956) in discrete time. Indeed, assuming no capital depreciation, it can be expressed by a first order difference equation in the capital-labour ratio $k_t = \frac{K_t}{L_t}$, that is

$$
k_{t+1} = \frac{\sigma\, f(k_t)}{1+\lambda} \tag{2.12}
$$

where σ is the saving ratio assumed as constant, $f(\cdot)$ is the production function and λ is the natural growth rate of population.
By assuming that productivity is reduced by a so called pollution effect, Day introduces the following specification for f:

$$
f(k) = Ak^{\beta}(m-k)^{\gamma} \tag{2.13}
$$

where $A > 0$ is a productivity parameter, $\beta > 0$ is the elasticity of capital, $m > 0$ is the state of environment if private production is not performed and $\gamma > 0$ weighs the effects of pollution.
Then, the map that describes the dynamics of the model becomes

$$
k_{t+1} = \frac{\sigma\, Ak_t^{\beta}(m-k_t)^{\gamma}}{1+\lambda}. \tag{2.14}
$$

The map in (2.14) is unimodal, C^1, and admits a unique maximum point $\overline{k} = \left(\frac{\beta\, m}{\beta+\gamma}\right)$, from which it can be obtained the maximum capital-labour ratio $k^m = f(\overline{k})$, that is

$$k^m = \frac{A\sigma}{1+\lambda}\beta^\beta\gamma^\gamma\left(\frac{m}{\beta+\gamma}\right)^{\beta+\gamma}. \tag{2.15}$$

In particular, note that the slope of the production function indefinitely grows as k approaches zero and consequently, for sufficiently small initial conditions $k_0 > 0$, growth is positive.
For values of A sufficiently small, the stationary state is monotonically achieved either from above or below. As A increases, the steady state capital stock increases until verify the equality $k^m = \overline{k} = k^s$, where k^s is the steady state capital-labour ratio. This represents the bifurcation point from which oscillations in levels of k occur for even higher values of A.
The condition $k^m \leq m$ is assumed to prevent negative values of k. Then, the author derives the following sufficient condition for the existence of growth cycles, that is

$$\frac{\beta}{\beta+\gamma}m < \frac{A\sigma}{1+\lambda}\beta^\beta\gamma^\gamma\left(\frac{m}{\beta+\gamma}\right)^{\beta+\gamma} \leq m. \tag{2.16}$$

Indeed, for a parameterisation satisfying such condition, the capital-labour ratio exhibits bounded oscillations (perhaps after a period of growth).
In order to determine if such cycles can be chaotic, Day shows the existence of a parameter set for the map such that the theorem in Li and Yorke (1975) is satisfied. For example, when $\beta = \gamma = m = 1$, the map (2.14) becomes

$$k_{t+1} = \frac{\sigma\, A}{1+\lambda}k_t(1-k_t) \tag{2.17}$$

from which, following Yorke and Yorke (1981), the existence of a parameter set such that

$$3.57 \leq \frac{\sigma\, A}{1+\lambda} \leq 4 \tag{2.18}$$

implies that irregular growth cycles appear (See panels (a), (b), (c) and (d) in Figure 2.1).

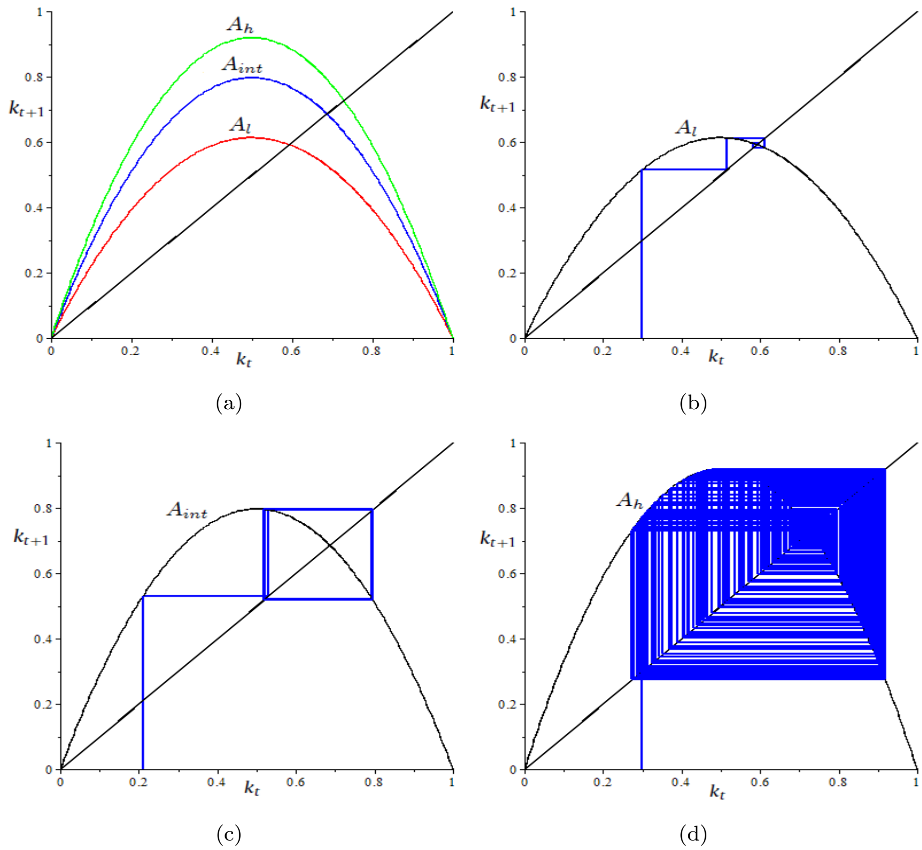


Figure 2.1: Parameter set: $\sigma = 0.5, \lambda = 0.02, A_l = 5, A_{int} = 6.5, A_h = 7.5$. (a) Changes in the shape of the map, as A varies. (b) Convergence to the fixed point for $A = A_l$ ($k_0 = 0.3$). (c) Convergence to a 2-cycle for $A = A_{int}$ ($k_0 = 0.21$). (d) Chaotic regime for $A = A_h$ ($k_0 = 0.3$).

2.3.2 The Zhang's (1999) model

In this model, the author considers an economy à la John and Pecchenino (1994), in which at every period t there are two overlapping generations of individuals. The representative agent of each generation gets utility from consumption, c_{t+1}, and environmental quality, E_{t+1}, when old. The utility function $U(c_{t+1}, E_{t+1})$ is assumed to be increasing with respect to each argument and strictly concave. Furthermore, some Inada like conditions are satisfied, in order to get an optimal boundle with strictly positive values for c and E.
The individual supplies inelastically a unit of labour only in the first period, earning a real wage w_t, and optimally allocate w_t between private savings s_t[9]

[9] Savings are inelastically supplied to the firms.

for the old age consumption and environmental improvement expenditures, m_t. In the old age, the agent earns $(1 + r_{t+1} - \delta)s_t$, where r and δ are the real rate of return and the capital depreciation rate, respectively. Then, the individual life-cycle budget constraints are given by:

$$w_t = s_t + m_t;\ c_{t+1} = (1 + r_{t+1} - \delta)s_t. \tag{2.19}$$

The environmental quality evolves according to the following rule, staken from John and Pecchenino (1994):

$$E_{t+1} = (1 - b)E_t - \beta\, c_t + \gamma\, m_t \tag{2.20}$$

where $b \in (0, 1)$ represents the degree of autonomous evolution of environmental quality, $\beta\, c_t$ measures the consumption degradation of the environment, and $\gamma\, m_t$ measures the environmental improvement (β and γ are assumed to be positive). The production function $f(k_t) : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be C^2 and strictly concave with respect to k_t (the capital-labour ratio). Firms maximise profits and then

$$w_t = f(k_t) - k_t f^{'}(k_t),\ r_t = f^{'}(k_t). \tag{2.21}$$

The representative agent maximises his utility with respect to (2.19) and (2.20). Then, the first order necessary and sufficient condition for the maximisation problem reads as

$$(1 + r_{t+1} - \delta)U_c(c_{t+1}, E_{t+1}) - \gamma\, U_E(c_{t+1}, E_{t+1}) = 0. \tag{2.22}$$

Hence, a perfect foresight competitive equilibrium is characterised by the equation (2.21), the condition (2.22) and the market clearing condition $k_{t+1} = s_t$.
To simplify the analysis, the author assumes a constant elasticity of substitution between consumption and environment, η_E, defined as:

$$\eta_E \equiv \frac{E}{c}\frac{U_E}{U_c} > 0, \tag{2.23}$$

and this allows to rewrite equation (2.22) as

$$c_{t+1} = \frac{E_{t+1}[1 + f^{'}(k_{t+1}) - \delta]}{\eta_E \gamma}. \tag{2.24}$$

From the market clearing condition, the author gets the following relationship between capital and environment

$$k_{t+1} = \frac{E_{t+1}}{\eta_E \gamma} \tag{2.25}$$

and then the study of the dynamics can be reduced to the analysis of the following first-order difference equation:

$$E_{t+1} = \frac{\eta_E}{1+\eta_E}\left(\left[1 - b - \frac{\beta(1-\delta)}{\eta_E \gamma}\right] E_t + [\gamma(1-\alpha(k_t)) - \beta\alpha(k_t)] f(k_t)\right) \quad (2.26)$$

where $\alpha(k) \equiv \frac{kf'(k)}{f(k)}$ is the capital share of output. In order to simplify the analysis, α is assumed as constant. Therefore, by considering $f(k_t) = Ak_t^{\alpha}$ (where $A > 0$ represents the productivity parameter), Zhang gets the expression

$$E_{t+1} = a_0 E_t + \left[\frac{A\eta_E^{1-\alpha}[\gamma(1-\alpha) - \beta\alpha]}{\gamma^{\alpha}(1+\eta_E)}\right](E_t)^{\alpha} \equiv G(E_t) \quad (2.27)$$

where $a_0 \equiv \frac{(1-b)\eta_E\gamma - \beta(1-\delta)}{\gamma(1+\eta_E)}$. It can be noticed that this parameter is less than 1 and it is independent with respect to α.
About the dynamics of the model, an interior fixed point exists if and only if $\gamma(1-\alpha) - \beta\alpha > 0$. In particular, Zhang shows that, when $a_0 \geq 0$, the unique stationary equilibrium is given by

$$E^* = \left[\frac{A\eta_E^{1-\alpha}[\gamma(1-\alpha) - \beta\alpha]}{(1-a_0)\gamma^{\alpha}}\right]^{\frac{1}{1-\alpha}} \quad (2.28)$$

and the following proposition holds:

Proposition 1 *Suppose $\gamma(1-\alpha) - \beta\alpha > 0$. Then, if $a_0 \geq 0$, map G is monotonically increasing; for all $E_0 \in (0, +\infty)$, $\lim_{t \to +\infty} G^t(E_0) = E^*$. That is, there exists a unique and asymptotically stable (attracting) positive steady state.*

When $a_0 < 0$, dynamics may exhibit complexity. By assuming $-\frac{\alpha^{-\frac{\alpha}{1-\alpha}}}{1-\alpha} < a_0$, Zhang proves that if $a_0 \in [-\frac{1+\alpha}{1-\alpha}, 0)$, E^* is stable. Instead, if a_0 decreases in the interval $\left(-\frac{\alpha^{-\frac{\alpha}{1-\alpha}}}{1-\alpha}, -\frac{1+\alpha}{1-\alpha}\right)$, the map G exhibits the classical period-doubling sequence, summarised by the bifurcation diagram in Figure 2.2. It can be noticed that A does not play any role in the dynamic properties of the model: it only influences the level of E at the stationary state.

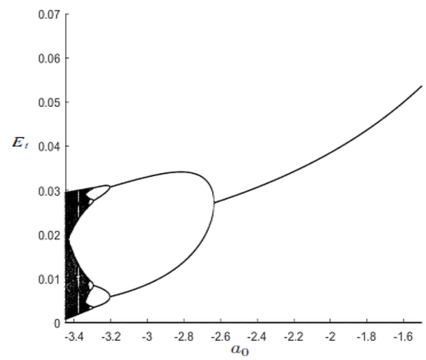


Figure 2.2: Occurrence of a period-doubling sequence as a_0 decreases.

2.3.3 The Naimzada-Sodini (2010) model

In the case of discrete time unidimensional models with overlapping generations, a variation of the model proposed in Zhang (1999), with a more general production function, has been provided by Naimzada and Sodini. Remaining close to the paradigm introduced by John and Pecchenino, the authors propose a model in which a population of individuals is characterised by a utility function depending on the stock of an environmental good, E_t, and on the consumption of the private good, c_t. It is also assumed that E_t is negatively affected by the consumption but it is improved by specific environmental expenditures.

Differently from Zhang (1999), Naimzada and Sodini consider allocations in a decentralised economy. In this case, the choices of each agent between consumption and environmental expenditure generate externalities on the others. A consequence of this assumption is that the environmental maintenance, because of the environment is considered as a public good, is characterised by the classic free-riding problem. In particular, the functional form described by the authors is the following:

$$E_{t+1} = (1-b)E_t - \beta \sum_{i=1}^{N} c_t^i + \gamma \sum_{i=1}^{N} m_t^i + b\overline{E} \tag{2.29}$$

where $b \in (0,1)$ measures the autonomous evolution of environmental quality, $\beta > 0$ measures the consumption effect on environment, $\gamma > 0$ weighs the environmental expenditures efficiency, N is the number of the agents, and $\overline{E} > 0$ represent the long run value of the environmental index in absence of anthropic activity. Individual preferences are described by the utility function $U(c_{t+1}, E_{t+1})$, with U assumed as twice continuous and differentiable. Also in this case, Inada like conditions are also assumed to avoid corner solutions in the optimisation problem.

On the production side, compared with Zhang (1999), the model considers a more general specification, that is a CES-technology:

$$Y = Af(k_t) = A\big(\alpha\, k_t^{-\rho} + (1-\alpha)\big)^{-\frac{1}{\rho}} \tag{2.30}$$

where k_t is the physical capital level at t, $A > 0$ is a scaling parameter, $\alpha \in (0,1)$ measures the degree of capital intensity of production, and $\theta = \frac{1}{1+\rho}$ represents the elasticity of substitution between labour and capital, with $\rho > 0$. From the optimality conditions, equilibrium expressions of wage rate w_t, and interest rate r_t are derived as

$$w_t = A(1-\alpha)\big(\alpha\, k_t^{-\rho} + (1-\alpha)\big)^{-\frac{1+\rho}{\rho}}, \tag{2.31}$$

$$r_t = A\alpha\, k_t^{-(1+\rho)}\big(\alpha\, k_t^{-\rho} + (1-\alpha)\big)^{-\frac{1+\rho}{\rho}}. \tag{2.32}$$

By assuming that agents are identical, the problem faced by the individual born at t is:

$$\max_{c_{t+1}, m_{t+1}} U(c_{t+1}, E^e_{t+1})$$

in which

$$E^e_{t+1} = (1-b)E_t - \beta\, Nc_t + \gamma(m^i_t + (N-1)m^e_t) + b\overline{E} \tag{2.33}$$

represents the expected environmental quality at time $t+1$ (depending on m^e_t, that is, the expectation of agent i about strategies of others $N-1$ identical agents) and the following constraints apply:

$$w_t = s_t + m_t, \tag{2.34}$$

$$c_{t+1} = (1 + r_{t+1} - \delta)s_t, \tag{2.35}$$

where s_t is the saving and δ represents the depreciation rate of capital. By assuming that the young individual at t is able to perfectly foresee the environmental index E_{t+1} and recalling the equations(2.31)-(2.32), the equilibrium conditions for all t become:

$$-U_{c_{t+1}}(\cdot,\cdot)(1 + r_{t+1} - \delta) + \gamma\, U_{E^e_{t+1}}(\cdot,\cdot) = 0, \tag{2.36}$$

$$m^e_t = m^*_t, \tag{2.37}$$

$$E_{t+1} = (1-b)E_t - \beta\, Nc^*_t + \gamma\, Nm^*_t + b\overline{E}, \tag{2.38}$$

$$k_{t+1} = s^{i*}_t. \tag{2.39}$$

In order to analyse dynamics of the model, the authors, as in Zhang (1999), introduce a constant elasticity of substitution between consumption and environment, defined as

$$\eta_E \equiv \frac{EU_E}{cU_c} > 0. \tag{2.40}$$

Hence, they characterise the intertemporal equilibrium conditions by means of a nonlinear difference equation in E_t:

$$E_{t+1} = \frac{\eta_E}{N+\eta_E}\Bigg[\left((1-b) - \frac{N\beta(1-\delta)}{\gamma\eta_E}\right)E_t + NA\left[\gamma(1-\alpha) - \beta\alpha\left[\frac{E_t}{\gamma\eta_E}\right]^{-\rho}\right] \\ \times\left[\alpha\left(\frac{E_t}{\gamma\eta_E}\right)+1-\alpha\right]^{-\frac{1+\rho}{\rho}} + b\overline{E}\Bigg] \equiv Z(E_t). \tag{2.41}$$

Therefore, Naimzada and Sodini describe the dynamics distinguishing between two possible cases, depending on the sign of ρ: (i) $\rho < 0$ and (ii) $\rho > 0$. In the case (i), the authors show that the map admits a unique positive fixed point and highlights that qualitative results similar to the ones in Zhang (1999) are achievable. Then, the unique fixed point E^* could be attracting or repelling and, in the latter case, limit cycles or a chaotic attractor arises. However, differently from Zhang (1999), the scaling parameter A affects the stability of E^*. Indeed, by starting from the parameterisation $\alpha = 0.831, \beta = 0.55, \gamma = 1.56, \delta = 0.001, \eta_E = 5, \rho = -0.4, A = 3, b = 0.52, \overline{E} = 1, N = 2000$, an increase in A generates a loss of stability of E^* through a period-doubling bifurcation. In addition, for $A \simeq 6$, the map exhibits chaotic dynamics.

In the case (ii), the map admits an odd number of fixed points and the ones with even index are unstable. Then, differently from (i), multiple equilibria may exist. In particular, the authors prove that, for $\rho >> 0$, three steady states exist.

An other interesting phenomenon shown by the authors is the following. By considering a parameter set for which a unique fixed point E_1^* exists, an increase of A is able to generate, through a fold bifurcation, two new steady states E_2^* and E_3^*, where $E_2^* < E_3^*$ is repelling and separates the basins of attraction of the two attracting fixed points E_1^* and E_3^*. Then, a change in A may be engine of a poverty trap (see Azariadis and Drazen (1990)).

Considering another configuration of parameters (see the parameter set in Figure 2.3), the authors provide another example in which a different sequence of dynamic phenomena occurs, as A varies.

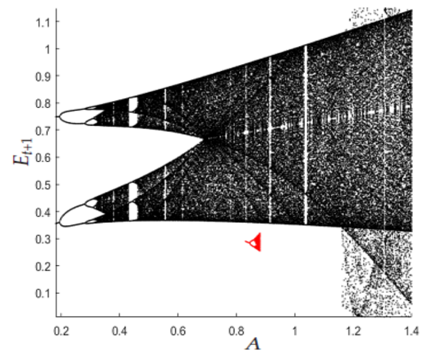


Figure 2.3: Parameter set: $\alpha = 0.07, \beta = 0.1, \eta_E = 11, \rho = 12, N = 100, \delta = 0.4, \gamma = 0.04, b = 0.58, \overline{E} = 22$. The part of the bifurcation diagram depicted in black is generated starting from $E_0 = 0.6$. The part depicted in red shows the existence of a second attractor for $A \in [0.83, 0.88]$ (the initial condition is $E_0 = 0.29$).

For A low, a unique repelling fixed point E^1, enclosed in an attracting limit 2-cycle, exists; by increasing A, the map undergoes the classic period-doubling sequence until a fold bifurcation generates two new fixed points, E^2 (repelling) and E^3 (attracting) with $E^3 < E^2 < E^1$. When A further increases, first E_3^* loses its stability through a flip bifurcation and then a cascade of flip bifurcations appears. Moreover, there exists a region of parameters for which coexistence of attractors occurs (see Figure 2.3). For larger values of A, the lower attractor dies and all the feasible trajectories are attracted by the remaining attractor that lives at the right of the repelling steady state E^2.
Figure 2.4 shows an interesting global phenomenon analysed by the authors, for even higher values of A: the remaining attractor enlarges and invades the space before occupied by the other attractor.

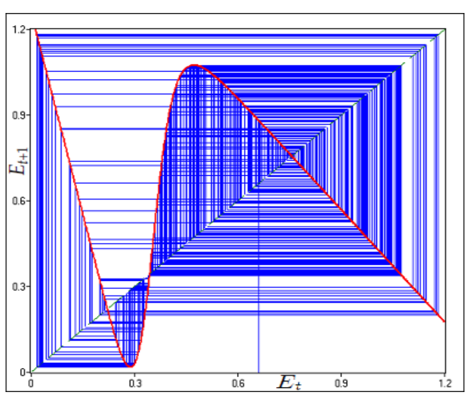


Figure 2.4: Merger of the attractors, $A = 1.21$.

Finally, an uncommon phenomenon analysed in the work is the following one. By fixing $\alpha = 0.07, \beta = 0.1, \eta_E = 11, \rho = 12, N = 100, \delta = 0.4, \gamma = 0.04, b = 0.58, \overline{E} = 22$, the second iterate of the map is characterised by two humps or more precisely by a maximum and a minimum point. By investigating the second iterate G^2, it can be noticed that an increase of A may lead to a fold bifurcation of G^2 inducing a stable 2-cycle and an unstable 2-cycle. In this case, the fold bifurcation arises far from the fixed point that maintains its instability for the whole process (see Figure 2.5).

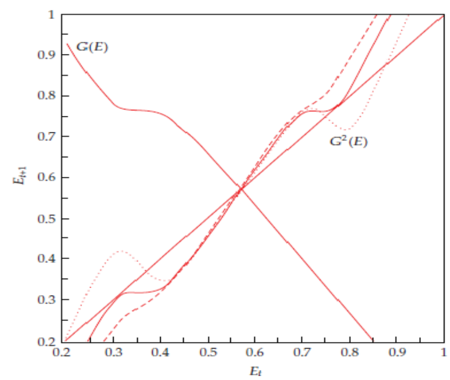


Figure 2.5: Figure reproduced with the permission of the authors. **Source**: Naimzada and Sodini (2010). Evolution of the second iterate of G (drawn at the bifurcation value $A = 0.132$). For $A = 0.10$, G^2 has no intersections with the main diagonal (dashed line); for $A = 0.132$, G^2 creates 2 tangent points (solid line) and for $A = 0.16$, two fixed points for G^2 are born (dotted line).

2.4 Global dynamics in an OLG model with productive open-access resources

In the OLG model proposed below, we assume that the economic activity depends on the exploitation of a free-access natural resource.[10]Models with a similar hypothesis on the productive sector are, for example, the aforementioned seminal work of Day (1982) or Le Kama (2001) that assumes, in a Ramsey model[11], a technology à la Day and characterises dynamics of the social planner's solution. In particular, the author detects the existence of a unique stationary equilibrium point, which is saddle-path stable.

Similarly to the approach that we propose, although in a continuous time framework, Antoci et al. (2011) analyse the dynamics of a decentralised Ramsey economy with a productive environment and where individuals do not consider how their own choices affect environmental dynamics, although they perfectly foresee its evolution. Both the work of Day (1982) and the work of Antoci et al. (2011) show that the saddle-path stability is not guaranteed, but the simultaneous presence of environmental dynamics and non-Pareto optimal allocations may generate fluctuations in both economic and environmental variables.

Surprisingly, the literature on the OLG framework developed a completely different paradigm. Indeed, it has focused on the analysis of models in which the environmental variable directly enters agents' utility function. This is the case, for instance, of the work of John et al. (1995), where agents behave as short-lived governments that devote a share of their resources to environmental defensive expenditures, and the work of Antoci and Sodini (2009), where the environmental dynamics is taken as exogenously determined and no positive environmental expenditures are introduced, although agents foresee the evolution of environmental quality.

As compared with the aforementioned works, another strand of literature assumed that the government takes charge of environmental expenditures, according to the idea that the environment is a public good. The government finances such expenditures through a share of a general taxation, as studied by Bovenberg and Heijdra (1998) and Heijdra et al. (2006).[12]

In the present model, we focus on the role of the environment as a productive input in a decentralised OLG framework and, in order to get readable results, we neglect the role of environment in providing free-access environmental goods for households.

With regard to the dynamics of the free-access natural resource, on one hand, we assume that production activity damages such resource and that agents consider it only as an externality (that is, no environmental expenditures are performed), as in Antoci et al. (2011). However, on the other hand, we assume that a government imposes a wage taxation to finance environmental maintenance, as in

[10]See Clark, 1976 for an in-depth discussion about how different kinds of natural resources may interact with economic activities.

[11]Saving is endogenous and the capital accumulation is bounded.

[12]See also Fodha and Seegmuller, 2012 for a paper that addresses the possible crowding-out effect between public and private expenditures for environmental maintenance.

Raffin and Seegmuller (2014).
In the analysis of the following model, we show how multiple equilibria may arise as well as complex dynamics, consistent with results provided in Zhang (1999), Lines (2005) and Naimzada and Sodini (2010) where the focus is on non-productive environmental goods. In addition, we prove the robustness of these results if a technology able to generate unbounded growth is assumed.[13]

2.4.1 The model

Individuals

We consider an overlapping generations economy characterised by individuals who live for two periods, young and old age. The time horizon is indexed by the discrete variable $t = 0, 1, 2, ..., \infty$. We assume no population growth and the size of each generation is normalised to one (Diamond, 1965). In his youth, each individual supplies inelastically the time endowment to the productive sector, remunerated at the real wage w_t, and shares his labour income between current consumption c_t and saving s_t for the old age.
Individual's preferences are defined on the consumption in young and old age, c_t and d_{t+1}, respectively, and on the level of basic services provided by public sector to individuals S_t. In particular, as in Naimzada et al. (2013) and Naimzada and Pireddu (2016), the utility function of the representative agent is given by:

$$U(c_t, d_{t+1}, S_t) = \ln(c_t) + \ln(S_t) + \phi \ln(d_{t+1}) \tag{2.42}$$

where $\phi > 0$ is the given discount factor. In addition, the government levies a tax on wage at the rate $0 < \tau < 1$. Saving, remunerated at the real interest factor R_{t+1}, is used to consume the final good in the old age. Hence, the consumer faces the following budget and feasibility constraints:

$$(1 - \tau)w_t = s_t + c_t \tag{2.43}$$

$$d_{t+1} = s_t R_{t+1} \tag{2.44}$$

$$c_t > 0, \ d_{t+1} > 0 \tag{2.45}$$

By using (2.43) and (2.44) one yields the life-cycle budget constraint

$$(1 - \tau)w_t = \frac{d_{t+1}}{R_{t+1}} + c_t \tag{2.46}$$

The maximisation of the utility function (2.42) under the intertemporal budget constraint (2.46) and the ex-post conditions $\overline{c}_t = c_t, \overline{d}_{t+1} = d_{t+1}$ define the

[13]This last result represents a novelty. Indeed, literature assessing the possibility of unbounded growth for capital (with an environmental productive input) has analysed models characterised by the existence of a unique saddle-path stable balanced growth path (see Gupta and Barman, 2009; Barman and Gupta, 2010).

following choices for the consumer, at the aggregate level:

$$c_t = \frac{(1-\tau)w_t}{1+\phi} \tag{2.47}$$

$$d_{t+1} = \frac{\phi R_{t+1}(1-\tau)w_t}{1+\phi} \tag{2.48}$$

By using (2.44) and (2.48), we obtain the following saving function:

$$s_t = \frac{\phi(1-\tau)w_t}{1+\phi}. \tag{2.49}$$

We can notice that, due to the use of a logarithmic specification for the utility function (2.42), the income and substitution effects cancel each other out. Therefore, saving does not depend on the interest rate.

Production

A unique material good is produced by a representative firm using a Cobb-Douglas technology:

$$Y_t = F(k_t, \overline{k}_t, E_t) = Ak_t^\alpha \overline{k}_t^\beta E_t^\gamma, \quad \text{with } \alpha, \beta, \gamma > 0 \quad \text{and } \alpha + \beta \leq 1 \tag{2.50}$$

where k_t is the physical capital, $\overline{k}_t$ represents the economy-wide average level of physical capital stock generating the production externality $\overline{k}_t^\beta$, $A > 0$ is a productivity scaling parameter and E_t is the stock of an open-access renewable natural resource (see Day, 1982; Antoci et al., 2011). Similar assumptions on production are considered both in growth models, such as the works of Smulders and Gradus (1996) and Brock and Taylor (2010), and in models regarding the optimal allocation and the use of natural resources (exaustible or renewable), such as fisheries economics (See Clark, 1976).

Concerning the inclusion of E_t in the production function, also many empirical studies have stressed the importance of the environment in affecting productivity. Environmental inputs are for example some kinds of natural resources, sink functions for pollution or land use (see Kozluk and Zipperer, 2015).

We note that, in one-sector growth models, the alternative assumptions on the elastic of capital $\alpha + \beta < 1$, $\alpha + \beta = 1$ characterise the impossibility or the possibility of unbounded growth (that is, a positive growth rate in the long run), respectively. In our work, the presence of an environmental sector may dramatically change the dynamic properties of the model, as it will be shown in the next sections.

In addition, by following the approach proposed in the literature on indeterminacy (see Cazzavillan, 2001),[14] we assume that the aggregate level $\overline{k}_t$ generates a positive production externality at time t.

[14] See John and Pecchenino (1994) for a different approach.

By assuming that the representative firm operates under perfect competition, it maximises the profit function

$$\pi(k_t) := Ak_t^\alpha \overline{k}_t^\beta E_t^\gamma - w_t - R_t k_t \tag{2.51}$$

taking $\overline{k}_t$, E_t, w_t and R_t as exogenously given. The maximisation yields the following equilibrium equations for wage and interest rate:

$$w_t = A(1-\alpha)k_t^\alpha \overline{k}_t^\beta E_t^\gamma \tag{2.52}$$

$$R_t = A\alpha k_t^{\alpha-1} \overline{k}_t^\beta E_t^\gamma. \tag{2.53}$$

Environmental Resource Dynamics

Taking into account that the production negatively affects the environmental resources[15] index E_{t+1} whereas the environmental public spending G_t increases its level, we introduce the following specification

$$E_{t+1} = \frac{\overline{E} + G_t^\theta}{1 + y_t^\lambda} \tag{2.54}$$

similar to the one introduced by Smulders and Gradus (1996)[16], and used by Antoci et al. (2016) in an OLG framework.[17]
In the equation (2.54), $\overline{E} > 0$ represents the value of the index when productive activity and public expenditures are null. The positive parameter λ weights the environmental influence of production whereas θ weights the positive impact of the public intervention on the free-access natural resource dynamics.
This specification, as compared with the linear form assumed by John and Pecchenino (1994), presents two advantages: first, from a mathematical point of view, it is effective to avoid negative values for the index in (2.54) (this is not consistent with the specification of the production function); second, it allows to consider possible nonlinear relations among economic activity, environmental dynamics and public expenditures (Rosser, 2001). This last point seems to be consistent with evidence of empirical analyses, as stated by Kozluk and Zipperer (2015).

[15] In the environmental literature both models with pollution dynamics and with environmental dynamics are considered. The choice between these two different approaches seems to be not crucial and therefore, according to the majority of works in OLG literature, we introduce an index measuring the quality of natural resources.

[16] The Smulders and Gradus (1996) specification, introduced to define the pollution stock P_t in continuous time, in our notation reads as $P_t = \frac{y_t^\lambda}{G_t^\theta}$. We prefer to use the expression in (2.54), rather than $E_{t+1} = \frac{G_t^\theta}{y_t^\lambda}$, in order to avoid the possibility that some parameterisations lead to two undesirable cases, that is (a) if production tends to zero, then the index (2.54) tends to $+\infty$; or (b) if environmental expenditures are zero, then production is equal to zero.

[17] Even if they neglect the existence of improving environmental expenditures.

Public Sector

The government aims to improve the stock of free-access environmental resources through investments intended for its maintenance and protection. This action is financed by a fixed share $\delta \in [0,1]$ of the general labour income taxation τw_t, that is:

$$G_t = \delta \cdot (\tau w_t), \tag{2.55}$$

whereas the residual part $S_t = (1-\delta)\cdot(\tau w_t)$ is earmarked for providing basic services for agents.

The assumption that environmental maintenance is performed by the government is in line with other theoretical works (see also Barman and Gupta, 2010). Moreover, it appears to be preferable to the two different frameworks provided in the literature: on one hand, the approach introduced by John and Pecchenino, in which the economy is not affected by externalities among contemporaries and agents are able to internalise the problem of optimizing environmental defensive expenditures;[18] on the other hand, the approach proposed in several works by Antoci and co-authors, in which no defensive expenditures are assumed.

2.4.2 Equilibrium Dynamics

This section is devoted to define and characterise the equilibrium dynamics of the model.

By considering the saving in (2.49); the market clearing condition in the capital market

$$k_{t+1} = s_t; \tag{2.56}$$

the production in (2.50); the wage in (2.52); the environmental public spending in (2.55); the aggregate consistency condition $k_t = \overline{k}_t$ and the dynamics of the free-access environmental resource in (2.54), we obtain the following two-dimensional system:

$$M : \begin{cases} k' = \frac{\phi(1-\tau)(1-\alpha)Ak^{\alpha+\beta}E^{\gamma}}{1+\phi} \\ E' = \frac{\overline{E}+\left(\delta\tau(1-\alpha)Ak^{\alpha+\beta}E^{\gamma}\right)^{\theta}}{1+(Ak^{\alpha+\beta}E^{\gamma})^{\lambda}} \end{cases} \tag{2.57}$$

where $'$ is the unit time advancement operator.

In order to avoid trivial dynamics on E, we introduce the following assumption:

Assumption 1 *If $\lambda = \theta$, then we assume* $\begin{vmatrix} \overline{E} & (\delta\tau(1-\alpha))^{\theta} \\ 1 & 1 \end{vmatrix} \neq 0$

Indeed, we note that if $\lambda = \theta$ and $\overline{E} = (\delta\tau(1-\alpha))^{\theta}$, the previous determinant is equal to zero, $E = (\delta\tau(1-\alpha))^{\theta}$ for any iteration and the dynamic

[18] Equilibrium dynamics are not Pareto optimal only because of the lack of coordination between generations.

equation for k boils down to $k^{'} = Bk^{\alpha+\beta}$, where $B := \frac{\phi(1-\tau)A(1-\alpha)^{1+\gamma\theta}(\delta\tau)^{\gamma\theta}}{1+\phi}$ and whose dynamical properties are well known.[19]

Proposition 2 *If $\theta > \lambda$ (resp. $\theta < \lambda$), then E' is U-shaped (resp. inverted U-shaped) with respect to Y.*

Proof. Taking into account the first derivative of E'

$$\frac{dE^{'}}{dY} = \frac{(Y^{\theta}((\theta-\lambda)Y^{\lambda}+\theta)(\delta\tau(1-\alpha))^{\theta} - \lambda\overline{E}Y^{\lambda})}{((1+Y^{\lambda})^2 Y)} \tag{2.58}$$

By direct calculations, we obtain that

$$sign\left(\frac{dE^{'}}{dY}\right) = sign(O) \tag{2.59}$$

where $O := \delta\tau(1-\alpha)^{\theta}Y^{\theta+\lambda}(\theta-\lambda) + (\delta\tau(1-\alpha))^{\theta}Y^{\theta}\theta - \overline{E}Y^{\lambda}\lambda$ and

$$sign(O) = (\delta\tau(1-\alpha))^{\theta}Y^{\lambda}\theta(\theta-\lambda) + (\delta\tau(1-\alpha))^{\theta}(\theta-\lambda)\theta. \tag{2.60}$$

It follows that E' may change its monotonic character at most one time. ■

By introducing the next change of variable:

$$Y = Ak^{\alpha+\beta}E^{\gamma} \tag{2.61}$$

we obtain the following one-dimensional map:

$$N : Y' = A\left(k^{'}\right)^{\alpha+\beta}(E')^{\gamma} = H(Y) \tag{2.62}$$

where $H(Y)$ is defined as follows

$$H(Y) := A\left(\frac{\phi(1-\tau)(1-\alpha)Y}{1+\phi}\right)^{\alpha+\beta}\left(\frac{\overline{E}+(\delta\tau(1-\alpha)Y)^{\theta}}{1+Y^{\lambda}}\right)^{\gamma} \tag{2.63}$$

Remark 1 *We underline that alternative hypotheses on some characteristics of the model, such as (a) assuming a direct taxation on firms (that is, post-tax disposable income is $(1-\tau)Ak_t^{\alpha}\overline{k}_t^{\beta}E_t^{\gamma}$ and $w_t = A(1-\alpha)(1-\tau)k_t^{\alpha}\overline{k}_t^{\beta}E_t^{\gamma}$) or (b) assuming that environment is negatively affected by comsumption (that is, $E_{t+1} = \frac{\overline{E}+G_t^{\theta}}{1+c_t^{\lambda}}$) and that agents behave as in Antoci et al. (2016) (that is, without considering effects of their action on environmental dynamics) lead to the same specification of map M, up to parameters rescaling.*

[19]If $\alpha+\beta < 1$, there exists a unique positive stationary equilibrium attracting every initial condition $k_0 > 0$. If $\alpha+\beta = 1$ and $B < 1$(resp. $B > 1$), dynamics starting from positive initial values of k converge to zero (positively diverge). If $\alpha+\beta > 1$, there exists a unique positive repulsive fixed point k^* and, for the initial condition $k_0 < k^*$(resp. $k_0 > k^*$), dynamics converges to zero (diverges to $+\infty$).

Remark 2 *We note that starting from a positive value of Y, the map generates positive (that is, economically relevant) values of Y for every iteration and for all the parameter sets.*

In what follows, we will characterise some properties of the map N. First of all, we can infer the relationship between fixed points of N and fixed points of the system M.

Proposition 3 *Let M, N be maps, defined in (2.63) and (2.57), respectively. Then, 0 is a fixed point for N if and only if $(0, \overline{E})$ is a fixed point for M.*

Proof. Let 0 be a fixed point for N. In order to obtain the result it is sufficient to notice that, by substituting $Ak^{\alpha+\beta}E^{\gamma} = 0$ in the map M, we get $k^{'} = k = 0$ and $E^{'} = E = \overline{E}$ for any iteration. The converse implication follows trivially. ■

Proposition 4 *Let M, N be maps, defined in (2.63) and (2.57), respectively. Then, $Y^* > 0$ is a fixed point for N if and only if there exist k^*, E^* such that (k^*, E^*) is a fixed point for M.*

Proof. Let Y^* be a fixed point for N. In order to obtain the result it is sufficient to notice that, by considering that $Ak^{\alpha+\beta}E^{\gamma} = Y^*$ for any iteration, we obtain $k^{'} = k = \frac{\phi(1-\tau)(1-\alpha)Y^*}{1+\phi}, E^{'} = E = \frac{\overline{E}+(\delta\tau(1-\alpha)Y^*)^{\theta}}{1+(Y^*)^{\lambda}}$. The converse implication follows trivially. ■

Corollary 1 *An attracting positive fixed point for N corresponds to an attracting positive fixed point for M, and vice-versa.*

Proposition 5 *Let M, N be maps, defined in (2.63) and (2.57), respectively. Then, we have the following cases:*

(a) If $\lambda < \theta$, $+\infty$ is a fixed point for N if and only if $(+\infty, +\infty)$ is a fixed point for M;

(b) If $\lambda = \theta$, $+\infty$ is a fixed point for N if and only if $(+\infty, (\delta\tau(1-\alpha))^{\theta})$ is a fixed point for M.

Proof. (a) Let $+\infty$ be a fixed point for N. For $\lambda < \theta$, we note that by considering $Ak^{\alpha+\beta}E^{\gamma} = +\infty$ for any iteration, we obtain $k^{'} = k = +\infty, E^{'} = E = +\infty$. Let us now assume that $(+\infty, +\infty)$ is a fixed point for M. Thus, from equations (2.61) and (2.63) we obtain $Y^{'} = Y = +\infty$.
(b) Let $+\infty$ be a fixed point for N. For $\lambda = \theta$, from $Ak^{\alpha+\beta}E^{\gamma} = +\infty$ we get $k^{'} = k = +\infty, E^{'} = E = (\delta\tau(1-\alpha))^{\theta}$. Let us now assume that $(+\infty, (\delta\tau(1-\alpha))^{\theta})$ is a fixed point for M. Thus, from equations (2.61) and (2.63) we obtain $Y^{'} = Y = +\infty$. ■

Remark 3 *In the previous Proposition, we avoided to discuss the case $\lambda > \theta$ since it would imply that $+\infty$ could not be a fixed point, as it will be shown in what follows.*

Definition of M in (2.57) implies that a n-period cycle for Y induces a cycle for k and E of periodicity less or equal to n. In particular, the following propositions can be stated:

Proposition 6 *If there exists a 2-period cycle for N, then there exists a 2-period cycle for E.*

Proof. Assume for contradiction that there exists a 1-period cycle for E and let Y_1 and Y_2 be values of a 2-period cycle for N. By writing $Y^{'}$ as

$$Y^{'} = m(Y)g(Y)^{\gamma}, \quad \text{with } g(Y) = E \tag{2.64}$$

We have that

$$\begin{cases} Y_2 = m(Y_1)g(Y_1)^{\gamma} = m(Y_1)E^{\gamma} \\ Y_1 = m(Y_2)g(Y_2)^{\gamma} = m(Y_2)E^{\gamma} \end{cases} \tag{2.65}$$

By solving both equations for E^{γ}, we obtain the following equality

$$\frac{Y_2}{Y_1} = \frac{m(Y_1)}{m(Y_2)}. \tag{2.66}$$

By considering that m is an increasing monotone function, we obtain a contradiction. Then, the result follows. ■

Proposition 7 *If there exists a $l-period$ cycle for N, then the trajectory of E associated with the cycle for N contains at least $\frac{l}{2}$ $\left(\frac{l+1}{2}\right.$, respectively$\left.\right)$ values, if l is even (odd, respectively).*

Proof. From Proposition 2 we may have that two different values of Y are associated with the same value of E. Then, the result directly follows. ■

The following subsections will focus on existence, multiplicity and stability of stationary equilibria. For both economic and mathematical reasons, we will consider cases $\alpha + \beta < 1$ and $\alpha + \beta = 1$ separately.

The case $\alpha + \beta < 1$

In this subsection, we consider the case $\alpha + \beta < 1$. In order to investigate whether unbounded trajectories exist or not, we have to take into account both the productivity of capital and evolution of E. In other words, the only condition $\alpha+\beta < 1$ is not sufficient to guarantee a capital accumulation process converging to a finite value.
As far as this is concerned, the following Proposition holds:

Proposition 8 *If $\alpha + \beta + \gamma(\theta - \lambda) < 1$, then $+\infty$ is not an attractor and all the attractors of the system are at finite distance.*

Proof. For Y high enough, we have:

$$H(Y) \simeq A\left(\frac{\phi(1-\tau)(1-\alpha)}{1+\phi}\right)^{\alpha+\beta}\left((\delta\tau(1-\alpha))^{\theta}\right)^{\lambda} Y^{\alpha+\beta+\gamma(\theta-\lambda)} \tag{2.67}$$

If $\alpha+\beta+\gamma(\theta-\lambda)<1$, $H(Y)$ definitely lies below the 45-degree line. By considering that the map is well-defined and continuous on the interval $[0,+\infty)$, we get the result. ■

Remark 4 *Note that, if the weight of public expenditure on the resource dynamics, θ, is lower than the impact of production, λ, the capital accumulation is always bounded. On the contrary, a sufficiently high value of the difference $\theta-\lambda$ may lead to an unbounded growth of k and E.*

Proposition 9 *If $\alpha+\beta<1$, then 0 is a repulsive steady state.*

Proof. If Y is small enough, we have

$$H(Y) \simeq A\left(\frac{\phi(1-\tau)(1-\alpha)}{1+\phi}\right)^{\alpha+\beta} \overline{E}^{\gamma} Y^{\alpha+\beta} \tag{2.68}$$

Therefore, $\lim_{Y\to 0^+} H^{'}(Y) = +\infty$ and the graph of $H(Y)$ starts above the 45-degree line. ■

Essentially, Proposition 9 states that, in the presence of decreasing social returns to scale with respect to the capital input, the economy cannot tend to an equilibrium with no positive activity.

In order to study the existence and multiplicity of positive stationary equilibria, we introduce the following function:

$$V(Y)=\frac{H(Y)}{Y}-1.$$

Stationary equilibria for $H(Y)$ are characterised by zeros of $V(Y)$. Function $V(Y)$ is continuous for $Y>0$ and $\lim_{Y\to 0^+} V(Y)=+\infty$ while $\lim_{Y\to+\infty} V(Y)=-1$ if $\alpha+\beta-1+(\theta-\lambda)\gamma<0$ and $\lim_{Y\to+\infty} V(Y)=+\infty$ if $\alpha+\beta-1+(\theta-\lambda)\gamma>0$. Therefore, a positive fixed point always exists if $\alpha+\beta-1+(\theta-\lambda)\gamma<0$ whereas if $\alpha+\beta-1+(\theta-\lambda)\gamma>0$ it is possible that a positive fixed point does not exist.

By direct calculation, we get

$$sign(V'(Y)) = sign(J(Y)) \tag{2.69}$$

where

$$J(Y)=\Big[(\alpha+\beta-1+(\theta-\lambda)\gamma)\,(\tau\ (1-\alpha))^{\theta}\, Y^{\theta+\lambda}+(\alpha+\beta-1+\theta\,\gamma)\,(\tau\ (1-\alpha))^{\theta}\, Y^{\theta}+ \\ +(\alpha+\beta-1-\gamma\,\lambda)\, Y^{\lambda}\overline{E}+(\alpha+\beta-1)\,\overline{E}\Big]$$

The following Lemmas classify some properties of $J(Y)$.

Lemma 1 *Assume $\lambda < \theta$. Then, the following cases arise:*

(a) If $\alpha + \beta < 1 - \theta\gamma$, then $J(Y)$ is always negative;

(b) If $1 - \theta\gamma < \alpha + \beta < 1 - (\theta - \lambda)\gamma$, then the equation $J(Y) = 0$ generically has (b1) zero or (b2) two solutions $\overline{Y}_1$ and $\overline{Y}_2$ with $\overline{Y}_2 > \overline{Y}_1$. In the case (b2) it follows that $J(Y) < 0$ for $Y \in (0, \overline{Y}_1) \cup (\overline{Y}_2, +\infty)$ and $J(Y) > 0$ for $Y \in (\overline{Y}_1, \overline{Y}_2)$;

(c) If $1 - (\theta - \lambda)\gamma < \alpha + \beta < 1$, then there exists a threshold value $\overline{Y}$ such that $J(Y) < 0$ for $Y \in (0, \overline{Y})$ and $J(Y) > 0$ for $Y > \overline{Y}$.

Proof. First, note that $\lim\limits_{Y \to 0^+} J(Y) = (\alpha+\beta-1)\overline{E} < 0$ and if $(\theta-\lambda)\gamma+\alpha+\beta-1 < 0$ (resp. $(\theta - \lambda)\gamma + \alpha + \beta - 1 > 0$) it follows that $\lim\limits_{Y \to +\infty} J(Y) = -\infty$ (resp. $\lim\limits_{Y \to +\infty} J(Y) = +\infty$). Then, by direct calculation we have that $sign(J'(Y)) = sign(X(Y))$ where

$$\begin{aligned} X(Y) = {} & ((\theta - \lambda)\gamma + \alpha + \beta - 1)(\delta\tau(1 - \alpha))^{\theta} Y^{\theta}(\theta + \lambda) + \\ & + (\gamma\theta + \alpha + \beta - 1)(\delta\tau(1 - \alpha))^{\theta} Y^{\theta - \lambda}\theta + (\alpha + \beta - 1 - \gamma\lambda)\lambda\overline{E} \end{aligned}$$

and

$$\begin{aligned} X'(Y) = {} & \frac{((\theta - \lambda)\gamma + \alpha + \beta - 1)(\delta\,\tau\,(1 - \alpha))^{\theta} Y^{\theta}\theta(\theta + \lambda)}{Y} + \\ & + \frac{(\alpha + \beta - 1 + \theta\,\gamma)(\delta\,\tau\,(1 - \alpha))^{\theta} Y^{\theta - \lambda}(\theta - \lambda)\theta}{Y}. \end{aligned}$$

The following cases arise: (a) If $\alpha + \beta < 1 - \theta\gamma$, we have that $J(Y)$ is a sum of negative terms for $Y > 0$. Therefore, $J(Y) < 0 \; \forall Y > 0$; (b) if $1 - \theta\gamma < \alpha + \beta < 1 - (\theta - \lambda)\gamma$, we have that there exists a point

$$\widetilde{Y} = \left(\frac{(\lambda - \theta)\,(\alpha + \beta - 1 + \theta\,\gamma)}{(\theta + \lambda)\,(\theta\,\gamma - \gamma\,\lambda + \alpha + \beta - 1)} \right)^{1/\lambda} \tag{2.70}$$

such that $X'(Y)$ is positive for $Y \in [0, \widetilde{Y})$ and negative for $Y > \widetilde{Y}$. This implies that $X(Y)$ may change its sign at most two times and the same result follows for $J(Y)$; (c) if $1 - (\theta - \lambda)\gamma < \alpha + \beta < 1$, we have that $X'(Y) > 0$ for $Y > 0$. Since $X(0) = \lambda\overline{E}\,(\alpha + \beta - 1 - \gamma\lambda) < 0$, $X(Y)$ can change its sign at most one time and the same result follows for $J(Y)$. ■

Lemma 2 *Assume $\lambda > \theta$. Then, the following cases arise:*

(a) If $\alpha + \beta < 1 - \theta\gamma$, then $J(Y)$ is always negative;

(b) If $1 - \theta\gamma < \alpha + \beta < 1$, then generically $J(Y) = 0$ has (b1) zero or (b2) two solutions, $\widehat{\overline{Y}_1}$ and $\widehat{\overline{Y}_2}$ with $\widehat{\overline{Y}_2} > \widehat{\overline{Y}_1}$. In the case (b2) it follows that $J(Y) < 0$ for $Y \in (0, \widehat{\overline{Y}_1}) \cup (\widehat{\overline{Y}_2}, +\infty)$ and $J(Y) > 0$ for $Y \in (\widehat{\overline{Y}_1}, \widehat{\overline{Y}_2})$.

Proof. First, note that $\lim_{Y\to 0^+} J(Y) = (\alpha+\beta-1)\overline{E} < 0$ and $\lim_{Y\to +\infty} J(Y) = -\infty$. Then, we observe that $sign(J'(Y)) = sign(X(Y))$ where

$$X(Y) = (\alpha+\beta-1-(\lambda-\theta)\gamma)(\delta\,\tau\,(1-\alpha))^{\theta}Y^{\lambda}(\theta+\lambda) + (\gamma\theta+\alpha+\beta-1)(\delta\,\tau\,(1-\alpha))^{\theta}\theta + \\ + (\alpha+\beta-1-\gamma\,\lambda)Y^{\lambda-\theta}\lambda\,\overline{E}$$

and

$$X^{'}(Y) = \frac{(\alpha+\beta-1-(\lambda-\theta)\gamma)(\delta\,\tau\,(1-\alpha))^{\theta}Y^{\lambda}\lambda\,(\theta+\lambda)}{Y} + \\ + \frac{(\alpha+\beta-1-\gamma\,\lambda)Y^{\lambda-\theta}(\lambda-\theta)\lambda\,\overline{E}}{Y}.$$

Therefore, (a) if $\alpha+\beta<1-\theta\gamma$, we have that $J(Y)$ is a sum of negative terms for $Y>0$. It follows that $J(Y)<0\ \forall Y>0$; (b) If $1-\theta\gamma<\alpha+\beta<1$, we have that $X'(Y)<0$ for $Y>0$, because $X(0) = (\alpha+\beta-1+\gamma\theta)\,(\delta\,\tau\,(1-\alpha))^{\theta}\,\theta>0$. Then, $X(Y)$ can change its sign at most one time and $J(Y)$ changes its sign at most two times. ■

Based on results of previous Lemmas, we have the following Propositions:

Proposition 10 *Assume $\lambda<\theta$. Then, the following cases arise:*

(1) If $\alpha+\beta<1-\theta\gamma$, then there exists a unique positive stationary equilibrium;

(2) If $1-\theta\gamma<\alpha+\beta<1-(\theta-\lambda)\gamma$ and the hypothesis in the case (b1) of Lemma 1 is satisfied, then there exists a unique positive stationary equilibrium;

(3) If $1-\theta\gamma<\alpha+\beta<1-(\theta-\lambda)\gamma$ and the hypothesis in the case (b2) of Lemma 1 is satisfied, then there exist threshold values $\overline{A},\underline{A}$, with $\underline{A}<\overline{A}$ such that for $A<\underline{A}$ or $A>\overline{A}$ the map admits a unique positive stationary equilibrium. For $\underline{A}<A<\overline{A}$ the map admits three positive stationary equilibria;

(4) If $1-(\theta-\lambda)\gamma<\alpha+\beta<1$, then there exists a threshold value $\widetilde{A}$ such that for $A<\widetilde{A}$ the map admits two positive stationary equilibria. For $A>\widetilde{A}$ the map does not admit any positive stationary equilibrium.

Proof. From results in Lemma 1, we can deduce that: (1) If $\alpha+\beta<1-\theta\gamma$, then $V(Y)$ is always decreasing and then the map $H(Y)$ admits a unique positive stationary equilibrium; (2) If $1-\theta\gamma<\alpha+\beta<1-(\theta-\lambda)\gamma$ and the case (b1) of Lemma 1, $V(Y)$ changes its sign one time and the map $H(Y)$ admits one positive stationary equilibrium; (3) if $1-\theta\gamma<\alpha+\beta<1-(\theta-\lambda)\gamma$ and the case (b2) of Lemma 1, $V(Y)$ changes its sign at most three times and the map $H(Y)$ admits at most three positive stationary equilibria; (4) If $1-(\theta-\lambda)\gamma<\alpha+\beta<1$, $V(Y)$ changes its sign at most two times and the map $H(Y)$ admits at most two positive stationary equilibria. ■

Proposition 11 *Assume $\lambda<\theta$. Then, the following cases arise:*

(a) If there exist three positive stationary equilibria $Y_1^*<Y_2^*<Y_3^*$, Y_2^* is always unstable;

(b) If $\alpha+\beta > \lambda\gamma$ and there exists a unique positive stationary equilibrium, it is globally asymptotically stable for every initial condition $Y > 0$;

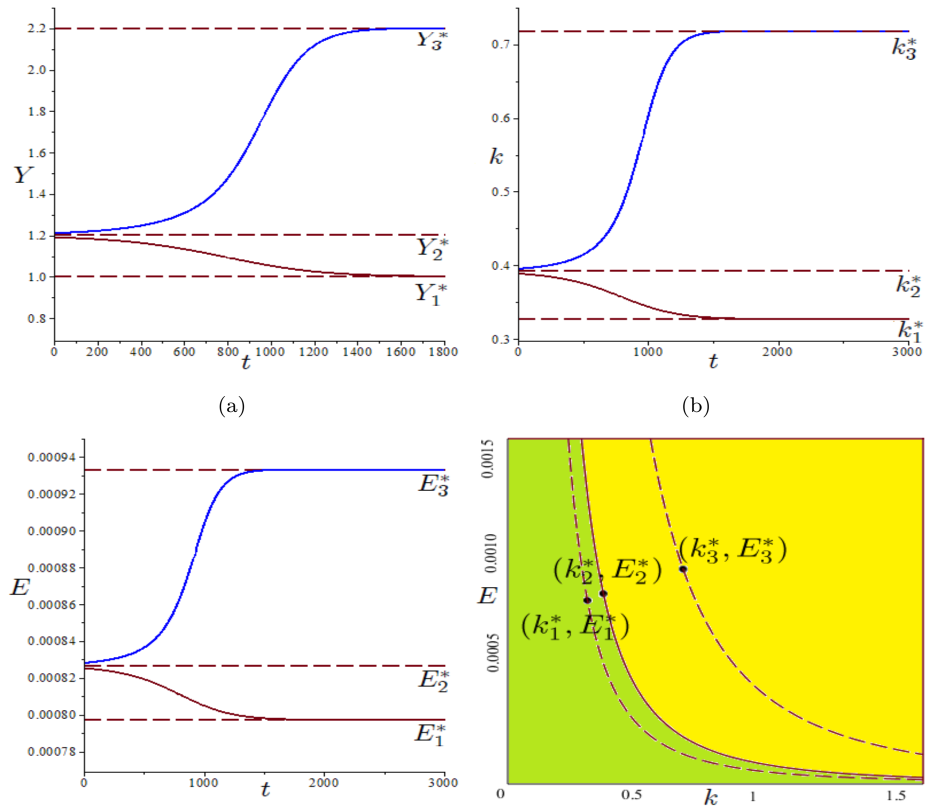
(c) If $\alpha+\beta > \lambda\gamma$ and there exist three positive stationary equilibria $Y_1^* < Y_2^* < Y_3^*$, Y_1^* and Y_3^* are locally asymptotically stable while Y_2^* is unstable.

Proof. The result in (a) follows by considering the behaviour of the map when Y approaches 0 and $+\infty$, implying that, at Y_2^*, the map intersects the 45-degree line from below. Results in (b) and (c) follow by the monotonicity of the map under the hypothesis $\alpha+\beta > \lambda\gamma$. ■

Remark 5 *We note that the required hypothesis $\alpha+\beta > \lambda\gamma$ in case (c) of Proposition 11 is not always verified when multiple equilibria exist. Failure of this hypothesis leads to a more complex situation which will be illustrated in section 4.*

Figure 2.6 shows coexistence of three stationary equilibria under the hypothesis (a) in Proposition 11. In this case, the map is monotone and, by starting from initial conditions lower or higher than Y_2^*, dynamics of N are captured by Y_1^* and Y_3^*, respectively. Moreover, from the associated time series of k and E, we note that the route to Y_3^* (starting from an initial value $Y \in (Y_2^*, Y_3^*)$) is associated with a growth in physical capital accumulation and in the stock of environmental resources.

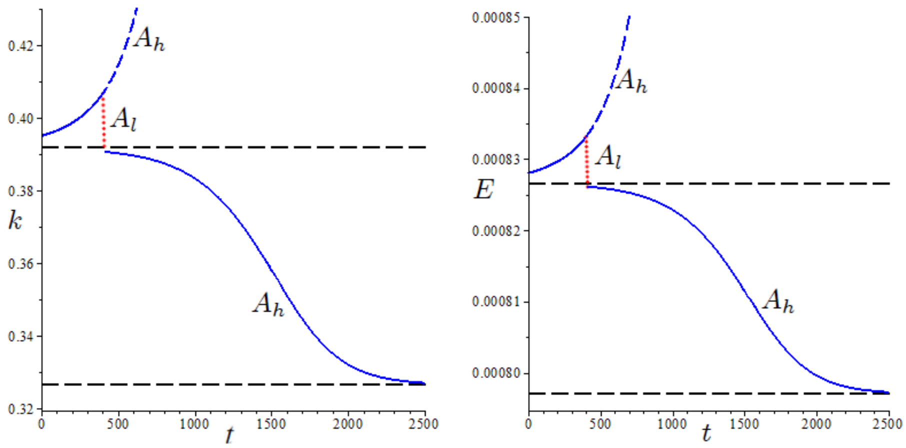
By investigating further properties of the relationship between M and N, it can be observed that starting from a positive stationary equilibrium Y^* for N, an invariant set in the plane (k, E), defined by $E = (\frac{Y^*}{Ak^{\alpha+\beta}})^{\frac{1}{\gamma}}$, is identified (see panel (d) in Figure 2.6). In particular, for this parameter set, $E = (\frac{Y^*}{Ak^{\alpha+\beta}})^{\frac{1}{\gamma}}$ describes the stable manifold on the saddle-point Y_2^* which separates the basins of attraction of Y_1^* and Y_3^*.



(a) (b) (c) (d)

Figure 2.6: Parameter set: $\alpha = 0.32, \beta = 0.6, \gamma = 0.4, \delta = 0.7, \theta = 3, \lambda = 2.87, \tau = 0.2, \phi = 0.6, \overline{E} = 0.0007, A = 48.6$. The map N has the following stationary equilibria: $Y_1^* = 1$, $Y_2^* = 1.2$, $Y_3^* = 2.2$. The map is monotone and then the unstable fixed point Y_2^* separates the basins of attraction of the two stable fixed points Y_1^* and Y_3^*. (a) Time series of Y converging to Y_1^* or to Y_3^* for different initial conditions; (b) Time series of k, (c) Time series of E; (d) Dashed lines describe iso-product curves in the plane (k, E) associated with stable fixed points of N, $Y_1^* = 1$ and $Y_3^* = 2.2$. The iso-product curve for the unstable $Y_2^* = 1.2$ (solid line) describes the stable manifold and divides the basin of attraction for Y_1^* (green region) and Y_3^* (yellow region), respectively.

By considering this parameter set and the results in Proposition 10, a decrease in A from $A_h = 48.6$ to $A_l = 48.3$ generates the disappearance of stationary equilibria Y_2^* and Y_3^*. Then, Y_1^* becomes the unique attractor of the system. In this case, a subsequent recovery of A to its original value A_h may not allow the restoration of the previous path, because the value of Y may have passed the threshold value Y_2^*. Figure 2.7 provides a numerical example of this occurrence, in terms of k and E:



(a) (b)

Figure 2.7: Parameter set: $\alpha = 0.32, \beta = 0.6, \gamma = 0.4, \delta = 0.7, \theta = 3, \lambda = 2.87, \tau = 0.2, \phi = 0.6, \overline{E} = 0.0007, A_h = 48.6, A_l = 48.3$. (a) Evolution of capital accumulation over time when a shock in A occurs; (b) Evolution of the stock of environmental resources over time when a shock in A occurs. The red pointed vertical line represents the temporary (exactly 10 iterations) time series generated by A_l.

Proposition 12 *Assume $\lambda > \theta$. Then, the following cases arise:*

(1) If $\alpha + \beta < 1 - \theta\gamma$, then there exists a unique positive stationary equilibrium;

(2) If $1 - \theta\gamma < \alpha + \beta < 1$ and the hypothesis in the case (b1) of Lemma 2 is satisfied, then there exists a unique positive stationary equilibrium;

(3) If $1 - \theta\gamma < \alpha + \beta < 1$ and the hypothesis in the case (b2) of Lemma 2 is satisfied, then there exist threshold values $\overline{A}, \underline{A}$, with $\underline{A} < \overline{A}$ such that for $A < \underline{A}$ or $A > \overline{A}$ the map admits a unique positive stationary equilibrium. For $\underline{A} < A < \overline{A}$ the map admits three positive stationary equilibria.

Proof. From results in Lemma 2, we can deduce that: (1) if $\alpha + \beta < 1 - \theta\gamma$, $V(Y)$ is always decreasing and then the map $H(Y)$ admits a unique positive stationary equilibrium; (2) If $1 - \theta\gamma < \alpha + \beta < 1$ and the case (b1) of Lemma 2, $V(Y)$ changes its sign one time and the map $H(Y)$ admits one positive stationary equilibrium; (3) If $1 - \theta\gamma < \alpha + \beta < 1$ and the case (b2) of Lemma 2, $V(Y)$ changes its sign at most three times and the map $H(Y)$ admits at most three positive stationary equilibria. ■

Figure 2.8 graphically shows the different results for the map N stated in Proposition 12, as A varies. In particular, it emphasises the role of A in inducing birth or disappearance of positive stationary equilibria.

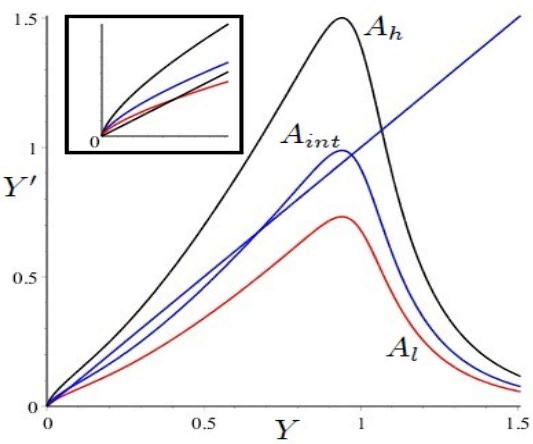


Figure 2.8: Parameter set: $\alpha = 0.3, \beta = 0.4, \gamma = 0.3, \delta = 0.1, \theta = 2.2, \lambda = 25.92, \tau = 0.31, \phi = 0.6, \overline{E} = 0.00001, A_l = 28.6, A_{int} = 38.6, A_h = 58.6$. Changes in the map shape as A varies. The enlargement highlights the behaviour of the map in a neighborhood of zero. For $A_l = 28.6$, the map admits a unique stationary equilibrium $Y^* = 0.0235$, for $A_{int} = 38.6$ the map admits three stationary equilibria $Y_1^* = 0.067$, $Y_2^* = 0.676$ and $Y_3^* = 0.97$, for $A_h = 58.6$ the map admits a unique stationary equilibrium $Y^* = 1.067$.

Proposition 13 *Assume $\lambda > \theta$. Then, the following cases arise:*
If there exist three stationary equilibria $Y_1^* < Y_2^* < Y_3^*$, Y_2^* is always unstable.

Proof. The result follows by considering the behaviour of the map when Y approaches 0 and $+\infty$, implying that over Y_2^*, the map intersects the 45-degree line from below. ■

Remark 6 *By considering results of Proposition 12 and Proposition 13, we observe that the existence of a unique stationary equilibrium for $H(Y)$ does not guarantee its asymptotic stability.*

The case $\alpha + \beta = 1$

This paragraph is devoted to deepen the particular instance of constant social returns to scale with respect to capital input. In this case, the map N becomes

$$H(Y) := A\left(\frac{\phi(1-\tau)(1-\alpha)Y}{1+\phi}\right)\left(\frac{\overline{E} + (\delta\tau(1-\alpha)Y)^\theta}{1+Y^\lambda}\right)^\gamma \tag{2.71}$$

Let

$$A^* = \frac{1+\phi}{E^\gamma\phi(1-\tau)(1-\alpha)} \tag{2.72}$$

As far as the existence and stability of stationary equilibria are concerned, the following propositions can be stated:

Proposition 14 *If $A < A^*$, then 0 is locally asymptotically stable; if $A > A^*$, then 0 is unstable.*

Proof. The result follows by considering that for $Y \to 0^+$ the map behaves as $\left(E^\gamma \frac{A\phi(1-\tau)(1-\alpha)}{1+\phi}\right) Y$. ■

Proposition 15 *If $\lambda < \theta$, $+\infty$ is an attractor; if $\lambda > \theta$, no unbounded trajectories for Y exist.*

Proof. The result follows by considering that $Y \to +\infty$ the map behaves as $A\left(\frac{\phi(1-\tau)(1-\alpha)}{1+\phi}\right) Y^{1+\gamma(\theta-\lambda)}$. ■

Proposition 16 (existence of positive stationary equilibria)

(1) If $\lambda < \theta$, there exist $\underline{A}$ and $\overline{A}$ where $\underline{A} := A^* < \overline{A}$ such that (i) for $A < \underline{A}$, a unique positive stationary equilibrium exists; (ii) for $\underline{A} < A < \overline{A}$, there exist two positive stationary equilibria and (iii) for $A > \overline{A}$, no positive stationary equilibria exist;

(2) If $\lambda > \theta$, there exist $\underline{A}$ and $\overline{A}$ where $\overline{A} := A^* > \underline{A}$ such that (i) for $A < \underline{A}$, no positive stationary equilibria exist; (ii) for $\underline{A} < A < \overline{A}$, there exist two positive stationary equilibria and (iii) for $A > \overline{A}$, there exists a unique positive stationary equilibrium.

Proof. By specializing $V(Y)$ in the case $\alpha + \beta = 1$, it is straightforward to obtain the result. ■

Some results on stability properties of interior stationary equilibria are derived as follows.

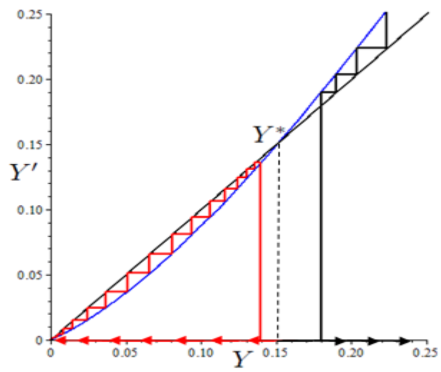
Proposition 17

(a) If $\lambda < \theta$ and there exists a unique positive stationary equilibrium Y^*, then Y^* is always unstable;

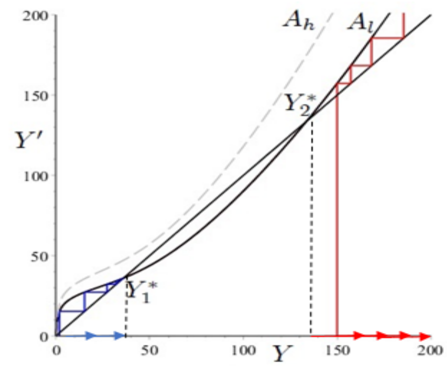
(b) if $\lambda > \theta$ and there exist two positive stationary equilibria Y_1^* and Y_2^* with $Y_1^* < Y_2^*$, then Y_1^* is always unstable.

Proof. By considering the way in which the map crosses the 45-degree line, we get the result. ■

Panels (a) and (b) in Figure 2.9 show the different scenarios in case (1) of Proposition 16. In particular, panel (a) shows that, given an initial value $Y_0 < Y^*$, in the long run, we obtain the destruction of the productive system and the environment returns to its pre-production state $\overline{E}$. Conversely, given an initial value $Y_0 > Y^*$, the path generated for Y induces an unbounded growth for both k and E. In panel (b), for $A = A_l$ (numerical values of this exercise can be found in the caption), we show two increasing trajectories leading to (i) a finite value of Y (and then to a finite value of k and E) and (ii) $+\infty$, respectively. Therefore, in the case $\alpha + \beta = 1$, two different initial conditions may lead to more dramatic differences in the economic growth path, compared with the case $\alpha + \beta < 1$. For $A = A_h$, we obtain the result that, for every initial condition $Y > 0$, the economy unboundedly grows.



(a)



(b)

Figure 2.9: (a) Parameter set: $\alpha = 0.6, \beta = 0.4, \gamma = 0.4, \delta = 0.1, \theta = 1.12, \lambda = 0.92, \tau = 0.21, \phi = 0.19, \overline{E} = 0.000001, A = 355$. For every initial condition $Y > 0$, the positive stationary equilibrium $Y^* = 0.153$ is unstable. Indeed, as shown by red and black arrows, the map admits only trajectories converging to 0 or $+\infty$. (b) Parameter set: $\alpha = 0.6, \beta = 0.4, \gamma = 0.4, \delta = 0.1, \theta = 3.12, \lambda = 1.92, \tau = 0.21, \phi = 0.19, \overline{E} = 0.1, A_l = 650, A_h = 850$. For $A = A_l$, the map admits two stationary equilibria $Y_1^* = 36.04$ and $Y_2^* = 133.62$. Blue arrows underline the convergence to Y_1^* while red arrows highlight the presence of an unbounded growth path for the economy. For $A = A_h$, no positive stationary equilibria are admitted (see the graph in grey) and all the trajectories (not depicted) positively diverge.

Figure 2.10 shows a numerical example of the case (2) in Proposition 16. In particular, (i) for $A = A_l$, the map does not admit any positive stationary equilibria and for every positive initial condition $Y > 0$ dynamics converge to 0; (ii) for $A = A_h$, the map admits two positive stationary equilibria Y_1^* (unstable) and Y_2^* (locally asymptotically stable). Therefore, an unbounded growth of the economy is not allowed because of the productive activity, that causes environmental damages excessive to be balanced by the public intervention. In other words, the economic growth path is bounded by the environmental depletion.

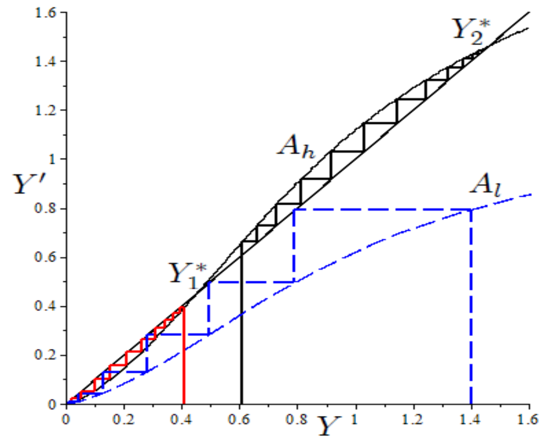
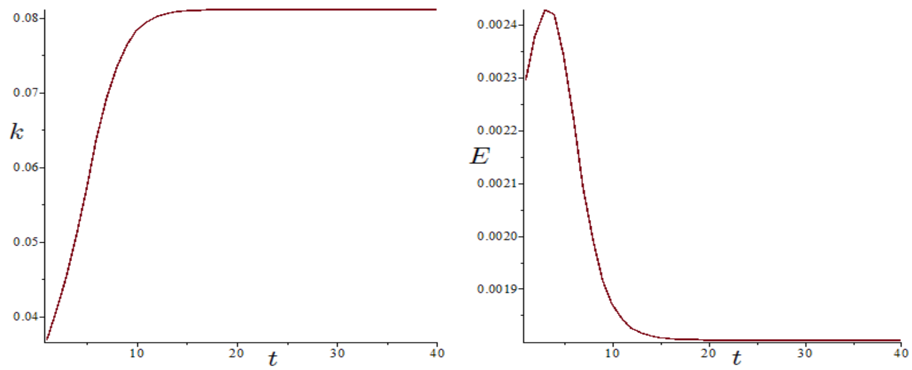


Figure 2.10: Parameter set: $\alpha = 0.6, \beta = 0.4, \gamma = 0.4, \delta = 0.1, \theta = 1.12, \lambda = 2.92, \tau = 0.21, \phi = 0.19, \overline{E} = 0.000001, A_h = 225, A_l = 125$. For $A = A_h$, the map admits two stationary equilibria $Y_1^* = 0.461$ and $Y_2^* = 1.458$. Two trajectories converging to zero or to the positive stationary equilibrium Y_2^* are depicted in red and black, respectively. For $A = A_l$, positive stationary equilibria disappear (see the graph in blue).

Panels (a) and (b) in Figure 2.11 show the associated time series of k and E: we note that the economic growth path leading to Y_2^* is associated with a growth in the physical capital accumulation as well as to an inverted U-shaped evolution in the stock of environmental resources.



(a)

(b)

Figure 2.11: Parameter set: $\alpha = 0.6, \beta = 0.4, \gamma = 0.4, \delta = 0.1, \theta = 1.12, \lambda = 2.92, \tau = 0.21, \phi = 0.19, \overline{E} = 0.000001, A = 225$. (a) Time series of k is monotonic increasing; (b) inverted U-shaped evolution of E.

In Figure 2.12, for $A = A_{int}$, we show another dynamic phenomenon generated by the map N when $\lambda > \theta$. In this context, the map is unimodal (different from the example in Figure 2.10 where N is monotone). This property, associated with the local stability of 0, induces the birth of a non-connected basin of attraction for 0 (grey lines on the x-axis highlight a portion of the basin of attraction). In particular, by choosing very high initial values, dynamics converge to 0.

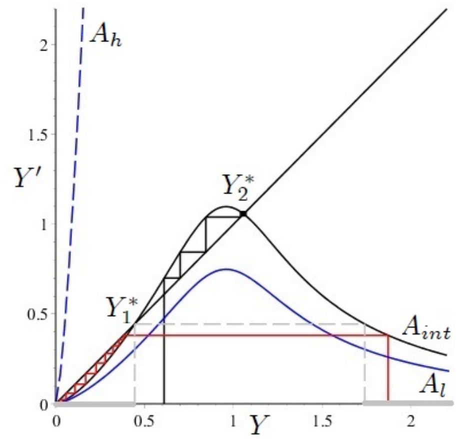


Figure 2.12: Parameter set:$\alpha = 0.6, \beta = 0.4, \gamma = 0.4, \delta = 0.1, \theta = 1.12, \lambda = 8.92, \tau = 0.21, \phi = 0.19, \overline{E} = 0.000001, A_h = 5000, A_{int} = 250, A_l = 150$. For $A = A_{int}$, the map admits two stationary equilibria $Y_1^* = 0.444$ (unstable) and $Y_2^* = 1.057$ (stable).

In the previous subsections, we analysed some properties of the map N assuming a given initial level Y_0. It is important to recall that a production level Y_0 can be generated by infinite combinations (k_0, E_0). This simple observation becomes relevant when multiple attractors exist, since for a given level k_0, two different levels E_0^1 and E_0^2 may lead to two different $\Omega-$limit set. Moreover, this property can have several consequences: for example, in Figure 2.6, for a given level k_0, a sufficiently low value of E_0 leads the economy to the stationary equilibrium Y_1^*, in which both E and Y assume low values, while a sufficiently high value of E_0 leads the economy to Y_3^*, in which both E and Y assume high values. On the contrary, in Figure 2.12, due to the non-invertibility of the map, for a given k_0, a sufficiently high value of E_0 generates dynamics leading to a stationary equilibrium with a lower value of Y.

Finally, it is worth to stress that an unbounded growth path of Y, associated with a permanent decrease of E, is possible only in the case $\lambda = \theta$.
Panel (a) in Figure 2.13 shows a numerical example with $\lambda = \theta$ in which Y positively diverges. Nonetheless, E decreases towards its stationary value (see panel (b)).

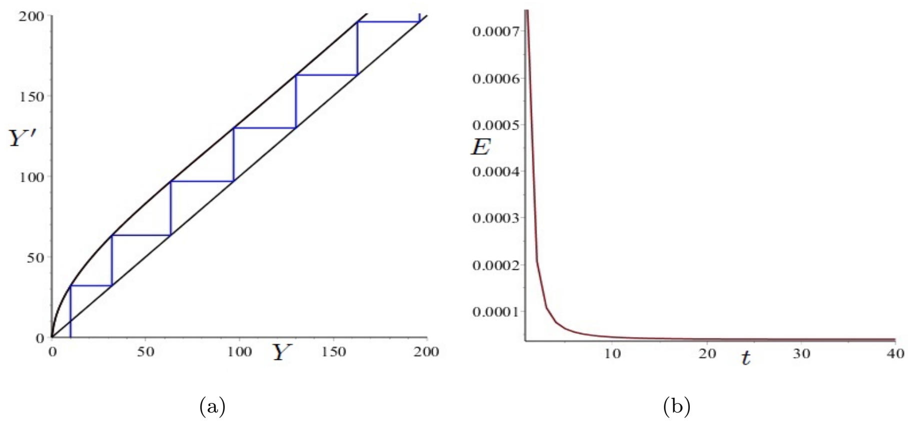


Figure 2.13: Parameter set: $\alpha = 0.6, \beta = 0.4, \gamma = 0.2, \delta = 1.1, \phi = 0.19, \lambda = 2.12, \theta = 2.12, \tau = 0.21, \overline{E} = 1.1, A = 150$. (a) The blue trajectory describes the diverging dynamics of N starting from an initial condition $Y_0 = 10$; (b) time series of E converges to the value $\underline{E} \simeq 0.00003976$.

2.4.3 The role of τ in the interplay between production and environment

By considering the parameter set $\alpha = 0.3, \beta = 0.2, \gamma = 5, \delta = 0.1, \phi = 0.19, \lambda = 2, \rho = 0.8, \sigma = 0.7, \theta = 0.4, \tau = 0.01, \overline{E} = 1, A = 1$, we let τ varies. The numerical exercise allows to observe the following phenomena: (i) for $\tau < 0.25$, an increase in general taxation is positively correlated with an increase in the production level and this is due to the positive effect of τ on the environmental variable E (that is a production input); (ii) for τ sufficiently high, a further increase in its value (and therefore the amount of public expenditure for environment) may decrease the level of environmental resources, due to the decrease of the total amount G devoted to the environmental defense (see Figure 2.14). Therefore, there exist values of τ for whom an increase in general taxation may increase (decrease) the stationary value of production (stock of environmental resources).

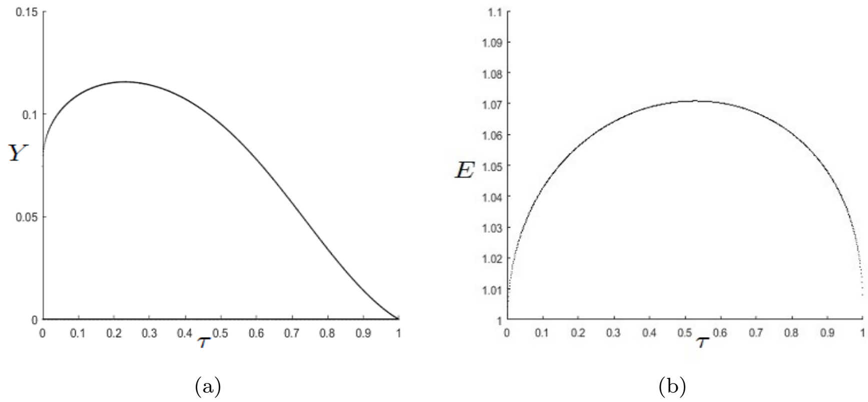
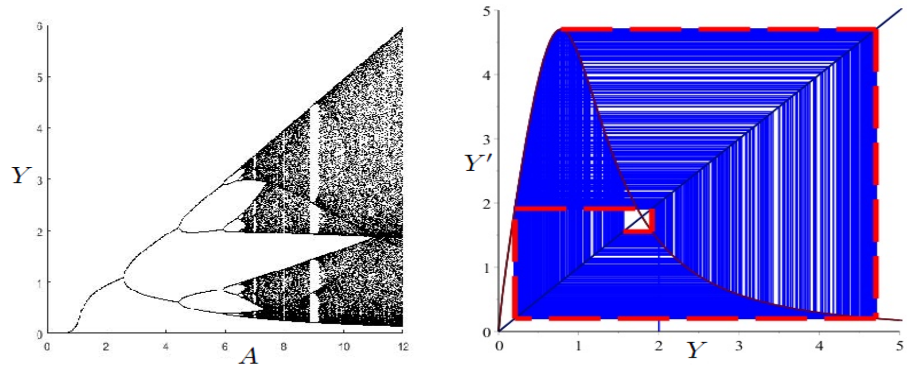


Figure 2.14: Parameter set: $\alpha = 0.3, \beta = 0.2, \gamma = 5, \delta = 0.1, \phi = 0.19, \lambda = 2, \rho = 0.8, \sigma = 0.7, \theta = 0.4, \tau = 0.01, \overline{E} = 1, A = 1$. Inverted U-shaped behaviour of Y and E with respect to the taxation rate τ.

2.4.4 Global Analysis

In the previous sections, we focused on (i) changes in the number of stationary equilibria as parameters vary (through saddle-node bifurcations) or (ii) changes in stationary equilibria levels and respective stability properties.
This section is devoted to investigating some economically relevant dynamic phenomena, which are related to the non-invertibility of the map.
More specifically, panels (a) and (b) of Figure 2.15 describe the case in which $\lambda > \theta$ and $\alpha + \beta < 1$ (the parameter set is specified in the caption). As the bifurcation diagram in Figure 2.15 (a) shows, for $A \in (0.47, 2.57)$ there exists a unique global attractor for any positive initial condition of Y. For $A \simeq 2.57$, the stationary equilibrium undergoes a flip bifurcation and an attracting 2-period cycle appears. By considering larger values of A, the classical sequence of period doubling bifurcations occurs until a chaotic attractor arises. According to Remark 2 and different from other OLG models (see Zhang, 1999)[20], trajectories are well-defined (stay positive) for all the parameter sets and for all the initial conditions. In panel (b), depicted for $A = 9.5$ we have highlighted the absorbing area, bounded by the critical point (in the Julia-Fatou sense) and its first three iterates, where long-run dynamics occur.
Therefore, the analysis shows that interplays between economic activity and productive environmental resources may cause complex dynamics.

[20]In this work, the initial value of the map cannot be taken too far from the stationary equilibrium.



(a) (b)

Figure 2.15: Parameter set: $\alpha = 0.25, \beta = 0.6, \gamma = 0.8, \delta = 0.5, \theta = .84, \lambda = 4, \tau = 0.1, \phi = 0.19, \overline{E} = 4$. (a) Bifurcation diagram of N with respect to A; (b) Absorbing area bounded by critical point and its first three iterations ($A = 9.5$).

Figure 2.16 stresses the ambiguous role of τ in modifying the dynamic properties of the model (the parameter set is specified in the caption). In particular, starting from the stable configuration of the system for $\tau = 0$, a rise in τ may induce a destabilization of the positive stationary equilibrium and a further increase in τ may be a source of complex dynamics. Nonetheless, for $\tau > \overline{\tau}$, the stationary equilibrium becomes stable again.

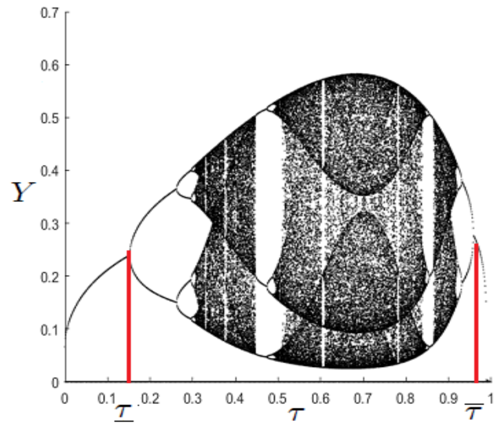
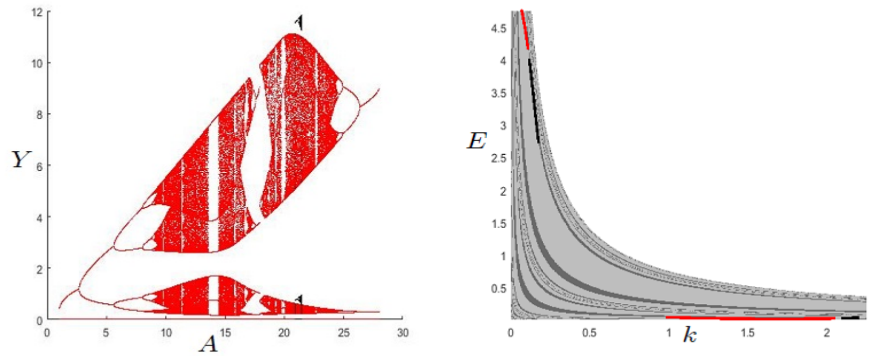


Figure 2.16: Parameter set: $\alpha = 0.3, \beta = 0.2, \gamma = 20, \delta = 0.1, \phi = 0.19, \lambda = 2, \theta = 0.4, \overline{E} = 1, A = 1$. For $\underline{\tau} < \tau < \overline{\tau}$, the stationary equilibrium is unstable; for $\tau > \overline{\tau}$ or $\tau < \underline{\tau}$ the stability is achieved.

Let us now consider the case $\alpha + \beta = 1$. Figure 2.17 shows a numerical example for which the map is bimodal (the parameter set is specified in the caption). For this parameterisation, the bifurcation diagram in Figure 2.17 (a) shows that the stationary equilibrium undergoes a flip bifurcation at $A = 2.67$ and an attracting 2-period cycle appears. By increasing A, we note a period doubling sequence leading to a chaotic regime, followed by a period halving sequence generating an attracting 2-period cycle for $A \in (26.34, 28.18)$. A further increase of A over the value 28.18 makes the interval $(0, Y_2^*)$ to lose its invariance property and then almost all trajectories positively diverge. Moreover, the graph highlights a phase of coexistence of two chaotic attractors (see also the basins of attraction in the plane (k, E) depicted in panel (b) of Figure 2.17).



(a) (b)

Figure 2.17: Parameter set: $\alpha = 0.4, \beta = 0.6, \gamma = 1.1, \delta = 0.9, \phi = 0.45, \lambda = 2.92, \theta = 12, \tau = 0.21, \overline{E} = 5$. (a) The bifurcation diagram with respect to A shows also the coexistence of two attractors for $A \in (20.92, 21.45)$; (b) basins of attraction (in light grey and dark grey, respectively) in the plane (k, E) when two attractors, depicted in red and black respectively, coexist ($A = 21.42$).

If the previous case shows that, as A varies, the change from a bounded to an unbounded growth for the economy arises after a phase of relative regularity of the dynamics (2-period cycle), the following example (see Figure 2.18) highlights that, as A increases, a direct transition from the chaotic regime to an unbounded growth for Y may occur.

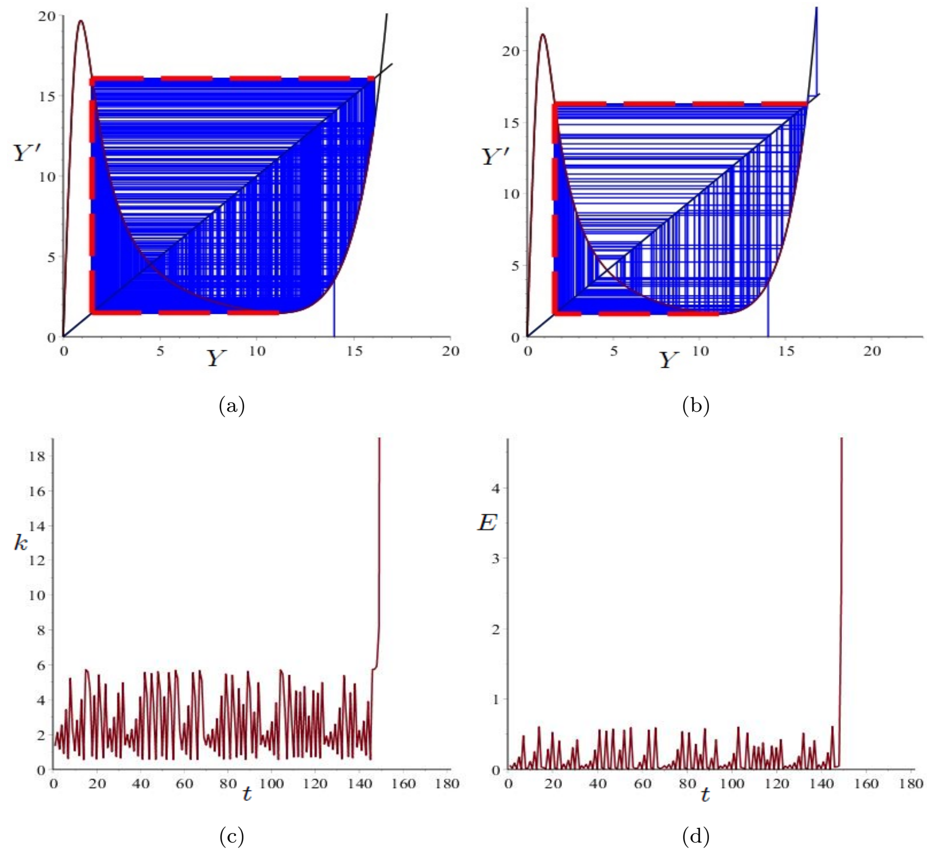


Figure 2.18: Parameter set: $\alpha = 0.3, \beta = 0.7, \gamma = 0.8, \delta = 0.4, \phi = 0.57, \lambda = 2.92, \theta = 15.12, \tau = 0.3, \overline{E} = 3$. (a) The graph describes the chaotic regime for $A = 40$, by starting from an initial condition $Y_0 = 14$; (b) the graph shows the image of the critical point Y_2^*. After a long transient, the trajectory positively diverges ($A = 43$); (c) time series of k associated to panel (b); (d) time series of E associated to panel (b).

The result dramatically changes when we assume $\lambda > \theta$. With regard to Figure 2.19, the parameter A plays a peculiar role: first, for $A \in (0, 536)$, 0 is the unique attractor of the system; second, for $A \in (536, 595)$, there exists an interior positive equilibrium for Y; third, an increase of A over the stability threshold ($A = 595$) induces a destabilization of the equilibrium and the birth of a chaotic attractor; finally, a further increase of A makes 0 the unique attractor of the system again.

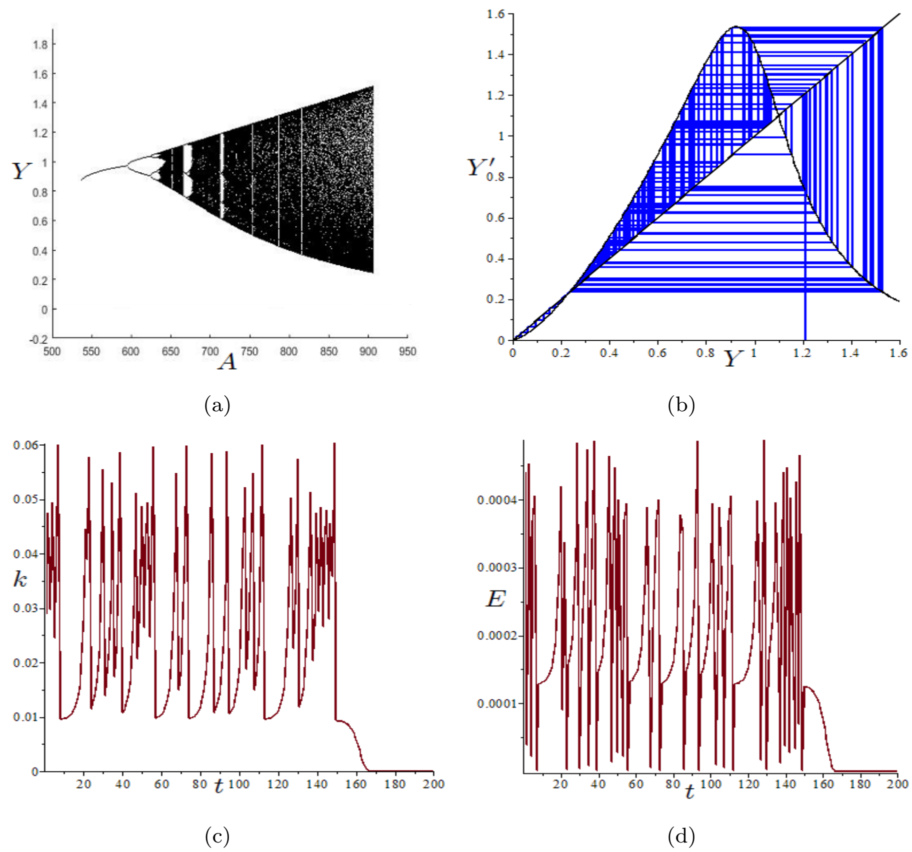


Figure 2.19: Parameter set: $\alpha = 0.82, \beta = 0.18, \gamma = 0.4, \delta = 0.07, \phi = 0.29, \lambda = 15.92, \theta = 1.12, \tau = 0.11, \overline{E} = 0.000001$. (a) Bifurcation diagram with respect to A; (b) behaviour of the map for $A = 920$. The set defined by the critical point (in the Julia-Fatou sense) and its first iterate has just lost its invariance property and all the trjectories converge to zero; (c) time series of k associated to panel (b); (d) time series of E associated to panel (b).

2.5 Conclusions

In this chapter, we have discussed the complex phenomena that the interplay between economic activity and the environment may generate. By reviewing the reference literature, we have shown how the dynamic systems governing the evolution of both economic and environmental variables are able to exhibit the occurrence of cyclical dynamics as well as multistability. The analysis of the model proposed in the previous section confirms the possible emergence of such phenomena and, in particular: (i) we noted that multiple attractors and chaotic dynamics may arise both in the case of efficient environmental expenditures and in the case of the dominant negative impact of production activity; (ii) we provided the condition under which an unbounded growth path for both production and environment may exist.

In order to describe possible further researches on nonlinear dynamics in the coevolution of economic and environmental systems, the heterogeneity in the modelling of environmental resources has not been investigated yet. These heterogeneities may concern the introduction of environmental variables affecting the system both as a production input and a consumption good, or the inclusion of a multiplicity of environmental resources both in the utility and production functions. The effects of this kind of heterogeneities on the dynamic properties in models such as those proposed in this chapter still represent open questions on which a scholar's agenda may concentrate upon.

Finally, a fruitful research question may concern the introduction of strategic interactions between agents. In particular, it could be relevant to analyse the dynamics driven by the recognition and the achievement of environmental targets, at the international level. Indeed, as described in Biancardi and Villani (2014), the decision process related to international environmental agreements may generate complex relationships (multistage games) which deserves further investigations.

Chapter 3

Nonlinear dynamics and global analysis of a heterogeneous Cournot duopoly model with differentiated products

3.1 Introduction

The purpose of this chapter is to discuss the role of horizontal product differentiation in a nonlinear Cournot duopoly in which firms adopt different decisional mechanisms.
Based on the pioneering work of Cournot (1838), in the last decades, several works have shown that oligopoly models may lead to complex behaviours, as described in Rand (1978), who analysed a simple Cournot duopoly in which nonlinear and non-monotonic firm reaction functions are assumed and discussed the arise of multiple equilibria as well as the possible emergence of random-like exotic effects in the resultant dynamics, in Shaffer (1984), who argued that chaotic duopoly dynamics depend on the assumption of *sophisticated* reaction functions[1], and in Puu (1991, 1996, 1998). In particular, Puu (1991) proposed a duopoly model with unimodal reaction functions obtained by solving the profit maximisation problem for the two firms. In the case of constant marginal costs, Cobb-Douglas preferences and isoelastic demand function, the author showed that the outputs of each firm can evolve through a sequence of period doubling

[1] In particular, Shaffer (1984) showed that chaotic behaviours may appear if firms account for phenomena such as inter-firm externalities. Instead, naive behaviours without such awareness rules out the emergence of chaotic phenomena.

bifurcations leading to chaos. Puu (1996, 1998) then extended such model to the case of three oligopolists and considered the problem of Stackelberg leadership where one of the firms, by taking into account the reaction functions of the rivals, becomes the leader.

As the firms have a perfect knowledge of the market demand, the decision mechanism analysed by Puu (1991) is the myopic best reply. By assuming that the rival does not change its decisions on the production, the firm solves its profit maximisation problem. Following these seminal works, the focus has shifted to less demanding requirements on the ability of firms in exploring the demand side of the market. In this regard, some works discussed the interaction between firms that have a biased knowledge of demand and that do not learn from such bias, continuing to systematically exploit, at every time period, the same strategy. Leonard and Nishimura (1999) considered a discrete time duopoly model with decreasing reaction functions in which the players have a misspecified knowledge of the demand function. They showed that, in this case, (i) a best replies process converges to a unique steady state that differs from the Nash equilibrium (obtained with full information) and (ii) this steady state may lose its stability as the bias in the knowledge of the demand function increases. The same dynamic results were confirmed by Bischi et al. (2004), in which the authors proposed a duopoly model à la Leonard and Nishimura (1999) where the assumption of decreasing reaction function is relaxed. Afterwards, several models in which duopolists are assumed to know their lack of information about the market demand and the decisions of the rival were developed. In this models, bounded rational firms adjust the previous decision through a decisional mechanism different from the classical best reply rule and, in particular, the literature has greatly deepened two mechanisms: (i) the gradient-like mechanism and (ii) the local monopolistic approximation (LMA hereafter).

The gradient-like approach describes firms that do not have a complete knowledge of demand and cost functions (see Bischi et al., 1999; Bischi and Naimzada, 2000). They use a local estimation of the marginal profit to update the production level. In particular, the output increases (decreases, respectively) if the marginal profit is positive (negative, respectively). A crucial role in defining the stability is then played by the parameter measuring the magnitude of this deviation in production, called speed of adjustment (see Cavalli and Naimzada, 2014; Cavalli et al., 2015).

In the LMA approach, proposed for the first time in Silvestre (1977), firms conjecture a linear demand function and estimate it through the current knowledge of the market in terms of quantities and price. Based on this estimate and assuming that competitors do not vary their production levels, the firm chooses the output that maximises the conjectured profit function (see Tuinstra, 2004; Bischi et al., 2007).

Recently, several authors have started to investigate the effects generated by the interaction of heterogeneous firms. Assuming firms that produce a *homogeneous* good, their analysis focused on the heterogeneity in the supply side of the market. In particular, Zhang et al. (2007) considered a duopoly model where (i) one firm has incomplete information and adopts the gradient-like mechanism whereas the rival has complete information and adopts the best reply and (ii) a nonlinear cost function is assumed. Tramontana (2010) discussed a duopoly model considering the same heterogeneous firms of Zhang et al. (2007) and introduced a nonlinearity in the demand function, instead of the cost function. In this work, the author showed how, by introducing a microfounded nonlinearity into the demand function, different routes to chaotic regimes may appear, depending on the value assumed by marginal costs and the speed of adjustment for the bounded rational firm. Cavalli and Naimzada (2014) and Cavalli et al. (2015)[2] then characterised the dynamic properties of a duopoly model in which firms are heterogeneous in decisional mechanisms and they both have restricted information on market demands (gradient-like approach vs. LMA).

Unlike the works mentioned above, whose analysis focuses on the supply side of the market, a strand of the literature focused on the study of the effects generated by product differentiation, or how the features related to the consumers' perception of goods may affect the duopolistic dynamics. To this purpose, this analysis has been carried out especially in Bertrand models where the product differentiation allows to overcome the so-called Bertrand paradox.[3] In particular, Ahmed et al. (2015), Brianzoni et al. (2015) and Gori and Sodini (2017) analyse the dynamics of a Bertrand duopoly where homogeneous decisional mechanisms and horizontal product differentiation are considered. In such models, the authors showed that the degree of product differentiation may play a destabilising role when it is set at too high or too low levels. Instead, Agliari et al. (2016) discussed the effects of product differentiation in a Cournot framework where firms adopt the same decisional mechanism (the gradient-rule) and showed that, unlike the literature on Bertrand models, only high levels of differentiation may have a destabilising effect on the system.

By removing the assumption that firms adopt the same decisional mechanism and remaining in a not overly demanding framework in terms of rationality and information, we consider a duopoly with nonlinear market demands for products of both varieties where firms (i) adopt respectively the gradient-rule and the LMA approach, and (ii) produce heterogeneous goods. In the analysis, we discuss how some relevant parameters (such as the speed of adjustment, the degree of differentiation and the marginal costs ratio) affect the stability of the Nash equilibrium and we show how the assumption of heterogeneous decisional mechanisms induces a partial change in the role played by the differentiation on the stability. Indeed, our investigation confirms the result shown in Agliari et al. (2016). Indeed, starting from a situation of stability for the Nash equilibrium, an increase in the differentiation destabilises the system, but we further show that also a low extent of product differentiation may be destabilising. From a dynamic point of view, we notice that a destabilisation may occur through Flip

[2] In Cavalli and Naimzada (2014) the authors consider the speed of adjustment as exogenous, i.e. independent on the level of production, while in Cavalli et al. (2015) it is taken as endogenous.

[3] The Bertrand paradox describes the situation in which a price war is waged between firms, leading the system on a state of perfect competition where the extra-profits of both firms are zero

and/or Neimark-Sacker bifurcations. Finally, we prove the existence of complex dynamics and the coexistence of attractors.
The economic intuition behind our analysis is the following: (i) if the degree of product differentiation is high, then the goods will tend to be independent and consequently, competition is less. In a context of isoelastic demands this implies that prices will react little to changes in quantities produced and are not able to bring the market back to a stationary equilibrium; on the other hand, (ii) if the degree of product differentiation is low, then the goods will tend to be indistinguishable and consequently, competition is high. this implies that prices will react excessively to changes in quantities produced and are not able to bring the market back to a stationary equilibrium.
The remainder of the chapter is organised as follows: Section 2 shows the main features of the static duopolistic game and proves the existence of the Nash equilibrium; Section 3 describes the decisional mechanisms of the firms; Section 4 refers to the local and global analysis of the model. Finally, Section 5 concludes.

3.2 The static model

We consider a duopoly market in which every firm i produces a differentiated good, whose prices and quantities are denoted by p_i and q_i, respectively, with $i \in \{1, 2\}$. Moreover, a continuum of identical consumers with preferences towards the two commodities q_1 and q_2 is assumed.
In particular, following Agliari et al. (2016), we determine the nonlinear demand functions from a monotonic transformation of a CES utility function (Mas-Colell et al., 1995) where the exponent is associated to the degree of product differentiation. Then, the utility function of the agents is

$$U(q_1, q_2) = q_1^\alpha + q_2^\alpha, \tag{3.1}$$

where $0 < \alpha \leq 1$[4] represents the degree of substitutability (differentiation) among the commodities. This utility function is maximised subject to the budget constraint

$$p_1 q_1 + p_2 q_2 = 1, \tag{3.2}$$

in which the consumers' income is assumed to be constant and equal to 1.
From the agents' allocative problem, the following inverse demand functions are derived:

$$p_1 = g_1(q_1, q_2) := \frac{q_1^{\alpha-1}}{q_1^\alpha + q_2^\alpha} \tag{3.3}$$

[4]For $\alpha = 0$, we notice that, from the consumer problem, any pair on the budget constraint is a solution of the optimization problem. This causes problems in defining demand functions. Ultimately, in a static context, this phenomenon generates a not interesting problem from an economic point of view, in the sense that the definition of supplies is irrelevant in the utility of the agents.

$$p_2 = g_2(q_1, q_2) := \frac{q_2^{\alpha-1}}{q_2^{\alpha} + q_1^{\alpha}}. \tag{3.4}$$

We notice that if $\alpha = 1$ the commodities are indistinguishable and consumers regard them as identical. Lower values of α makes the commodities as interchangeable and; furthermore, as α tends to zero, they become independent.
On the production side, the two duopolistic firms are characterised by a linear cost function given by

$$C_i(q_i) = c_i q_i \quad \text{with } i = 1, 2 \tag{3.5}$$

where c_i represent the positive constant marginal costs. Then, the expected profit function for the i-th firm is

$$\pi_i(q_i, q_j^e) = p_i(q_i, q_j^e)q_i - c_i q_i \quad \text{with } i, j = 1, 2;\ i \neq j \tag{3.6}$$

in which q_j^e is the expected output level of the rival.
Therefore, the unique Nash equilibrium of the Cournotian game can be derived (see Agliari et al., 2016):

Proposition 18 *The Nash equilibrium of the static Cournotian game is unique and it is given by*

$$E^* = \left(\frac{\alpha c_1^{\alpha-1} c_2^{\alpha}}{(c_1^{\alpha} + c_2^{\alpha})^2}, \frac{\alpha\, c_1^{\alpha}\ c_2^{\alpha-1}}{(c_1^{\alpha} + c_2^{\alpha})^2} \right).$$

3.3 LMA vs gradient learning

In an oligopolistic competition, the Nash equilibrium notion is based on the assumption that each firm knows what the rivals decide to do. In particular, each firm is assumed to know the entire demand curve for the good it produces. Then, the Nash equilibrium turns out to be highly demanding in terms of rationality and information. Indeed, it becomes interesting to investigate if the Nash equilibrium describes the long run behaviour of the market, that is if there exist mechanisms that, although they do not allow to achieve such equilibrium in one shot, lead to the Nash equilibrium at least asymptotically. In this case, we consider two different adjustment mechanisms requiring a low degree of rationality: the LMA approach and the gradient adjustment process.
To be precise, we assume that the firm 1 adopts the LMA approach, that is a bounded rational adjustment process based on the assumption that the firm has only a limited knowledge of the demand function (see Bischi et al., 2007; Naimzada and Tramontana, 2009; Cavalli et al., 2015). We assume that the firm 1 knows the market price, the output produced by the firm and the output produced by the rival at time t, that is $p_{1,t}$, $q_{1,t}$ and $q_{2,t}$ respectively. Moreover, the firm is able to get a correct estimate of the partial derivative $\frac{\partial g_1(q_{1,t}, q_{2,t})}{\partial q_{1,t}}$.
As in Bischi et al. (2007), the firm 1 conjectures that $q_{2,t+1} = q_{2,t}$ and a linear price function. Therefore, the expected price at time $t+1$ is

$$p^e_{1,t+1} = p_{1,t} + \frac{\partial\, g_1(q_{1,t}, q_{2,t})}{\partial\, q_{1,t}} (q_{1,t+1} - q_{1,t}) \tag{3.7}$$

from which, by considering the expression in (3.3), we get the following:

$$p^e_{1,t+1} = \frac{q_{1,t}^{\alpha-1}}{q_{1,t}^{\alpha} + q_{2,t}^{\alpha}} - \frac{\left((1-\alpha)\, q_{1,t}^{\alpha-2}\; q_{2,t}^{\alpha} + q_{1,t}^{2\,\alpha-2}\right)}{\left(q_{1,t}^{\alpha} +\; q_{2,t}^{\alpha}\right)^2} (q_{1,t+1} - q_{1,t})\,. \tag{3.8}$$

The output to produce at $t+1$ can be determined as the solution of the maximisation problem for the expected profit:

$$q_{1,t+1} = \underset{q_{1,t+1}}{\arg\max}[p^e_{1,t+1} q_{1,t+1} - c_1 q_{1,t+1}], \tag{3.9}$$

which leads to the equation[5]

$$q_{1,t+1} = \frac{1}{2}\left[\frac{\left(\frac{q_{1,t}^{\alpha-1}}{q_{1,t}^{\alpha}+q_{2,t}^{\alpha}} - c_1\right)(q_{1,t}^{\alpha} + q_{2,t}^{\alpha})^2}{q_{1,t}^{\alpha-2}\left[(1-\alpha)q_{2,t}^{\alpha} + q_{1,t}^{\alpha}\right]} + q_{1,t}\right] \tag{3.10}$$

Differently, we assume that the firm 2 adopts the gradient rule. In particular, we assume that the firm 2 does not have a global knowledge of the demand function, and tries to investigate how the market responds to its production changes through an empirical estimate of the marginal profit. This estimate may be obtained by market researches carried out at the beginning of the period t and then we assume that, although the firm is unaware of the market demand, it can obtain a correct empirical estimate of the marginal profit, $\frac{\partial \pi_2}{\partial q_2}$. With this type of information, the firm increases (decreases, respectively) its production if it perceives a positive (negative, respectively) marginal profit. We assume that the dynamic adjustment mechanism for the firm 2 reads as

$$q_{2,t+1} = q_{2,t} + k\frac{\partial \pi_2(q_{1,t}, q_{2,t})}{\partial q_{2,t}}, \tag{3.11}$$

where $k > 0$ represents the coefficient measuring the speed of adjustment of the output for firm 2 at time $t+1$ with respect to the marginal profit at time t.[6]

By taking into account expressions in (3.10) and (3.11), the two-dimensional system characterising the dynamics of the Cournot duopoly with differentiated products is the following:

[5] We notice that, for $\alpha = 0$, Equation 3.10 becomes $q_{1,t+1} = q_{1,t}(1 - c_1 q_{1,t})$, that defines dynamics converging to zero. This paradoxical result, typical of isoelastic demands, can be overcome by considering bounded demand functions (see Agliari et al., 2002).

[6] Trajectories of (3.11) may become negative. However, the analysis focuses only on initial values and parameters for which $q_{2,t}$ assumes a positive value.

$$M : \begin{cases} q_1^{'} = \frac{1}{2}\left[\frac{\left(\frac{q_1^{\alpha-1}}{q_1^{\alpha}+q_2^{\alpha}} - c_1\right)(q_1^{\alpha}+q_2^{\alpha})^2}{q_1^{\alpha-2}\left[(1-\alpha)q_2^{\alpha}+q_1^{\alpha}\right]} + q_1\right] \\ q_2^{'} = q_2 + k\left(\frac{\alpha\, q_2^{\alpha-1} q_1^{\alpha}}{\left(q_1^{\alpha}+q_2^{\alpha}\right)^2} - c_2\right) \end{cases} \tag{3.12}$$

where the symbol $'$ is the unit-time advancement operator and, as aforementioned, $\alpha \in (0, 1]$ while $k, c_1, c_2 > 0$. Due to the presence of a denominator with q_1 and q_2, we focus on dynamics which stay in the set FS for any iterations, where

$$FS = \{(q_1, q_2) : q_1 > 0, q_2 > 0\}. \tag{3.13}$$

Analogously to Agliari et al. (2016), we have the following result:

Proposition 19 *The Nash equilibrium E^* is a steady state of the system M described in (3.12). Contrariwise, the unique steady state of (3.12) is the Nash equilibrium.*

3.4 Dynamic properties of the model

In order to investigate the local stability of the Nash equilibrium, we consider the Jacobian matrix of the system (3.12), evaluated at E^*, $J_{E^*} =$

$$= \begin{bmatrix} -\frac{1}{2}\,\frac{(\alpha^2-3\,\alpha+2)c_1^{2\,\alpha} - c_2^{\alpha}(\alpha+4)(\alpha-1)c_1^{\alpha} + 2\,c_2^{2\,\alpha}}{((\alpha-1)c_1^{\alpha}-c_2^{\alpha})(c_1^{\alpha}+\ c_2^{\alpha})} & \frac{1}{2}\,\frac{c_1^{\alpha} c_2\,(c_1^{\alpha}-c_2^{\alpha})\alpha^2}{((\alpha-1)c_1^{\alpha}-c_2^{\alpha})(c_1^{\alpha}+c_2^{\alpha})c_1} \\ \frac{k c_1\, c_2\,(c_1^{\alpha}-c_2^{\alpha})(c_1^{\alpha}+c_2^{\alpha})}{c_1^{\alpha} c_2^{\alpha}} & -\frac{c_2^2 k(\alpha+1)c_1^{2\,\alpha} + (2\,c_2^2 k-\alpha)c_2^{\alpha} c_1^{\alpha} - c_2^{2\alpha+2} k(\alpha-1)}{c_2^{\alpha}\,\alpha\, c_1^{\alpha}} \end{bmatrix}.$$

The Nash equilibrium is locally asymptotically stable if the following Jury conditions (Elaydi, 2007) are satisfied:

$$\begin{cases} 1 - Tr(J_{E^*}) + Det(J_{E^*}) > 0 \\ 1 + Tr(J_{E^*}) + Det(J_{E^*}) > 0 \\ 1 - Det(J_{E^*}) > 0. \end{cases} \tag{3.14}$$

We can notice that the first condition in (3.14)

$$1 - Tr(J_{E^*}) + Det(J_{E^*}) = \frac{1}{2}\left[\frac{{c_2}^2 k({c_1}^{\alpha} + {c_2}^{\alpha})^2}{{c_2}^{\alpha}((1-\alpha){c_1}^{\alpha} + {c_2}^{\alpha})}\right] > 0 \tag{3.15}$$

is always fullfilled.

In order to determine the stability region of the Nash equilibrium in the space of parameters, in what follows we will characterise the boundary of such a region, defined by the equations

$$1 + Tr(J_{E^*}) + Det(J_{E^*}) = 0, \tag{3.16}$$

and

$$1 - Det(J_{E^*}) = 0. \tag{3.17}$$

By introducing the change of variable $x := \left(\frac{c_1}{c_2}\right)^\alpha$, if

$$h(\alpha, x) = \left(\alpha^2 + \frac{1}{4}\,\alpha - 1\right)x^2 - \left(\alpha^2 - \frac{5}{4}\,\alpha + 2\right)x + \alpha - 1, \tag{3.18}$$

$$z(\alpha, x) = \left(\alpha^2 + \frac{1}{2}\,\alpha - 1\right)x^2 - \left(\alpha^2 - \frac{3}{2}\,\alpha + 2\right)x + \alpha - 1 \tag{3.19}$$

are different from zero, both the relationships in (3.16) and (3.17) define k as function of α, c_1 and c_2, namely $F_l(\alpha, c_1, c_2)$ with $l = 1, 2$, respectively. Since these functions are both homogeneous of degree -2 with respect to c_1 and c_2, without loss of generality, (3.16) and (3.17) define the following functions $f_l : (\alpha, x) \to \widetilde{k}$ with $l = 1, 2$, respectively:[7]

$$f_1(\alpha, x) := \frac{1}{2}\left(\frac{\alpha\, x}{(x+1)^2}\right)\frac{-(\alpha^2 - 5\,\alpha + 4)x^2 + (\alpha^2 + 5\,\alpha - 8)x - 4}{h(\alpha, x)}; \tag{3.20}$$

$$f_2(\alpha, x) := \frac{1}{2}\left(\frac{\alpha\, x}{x+1}\right)^2 \frac{(1-\alpha)x + (\alpha+1)}{z(\alpha, x)} \tag{3.21}$$

where $\widetilde{k} = \frac{k}{c_2^2}$. We note that x is increasing with respect to $\frac{c_1}{c_2}$ and it varies in the interval $(0 + \infty)$.

The (i) shape of the graphs of these functions, (ii) their intersection points and (iii) the study of inequalities in (3.14) allow characterising the different dynamic properties of the Nash equilibrium and the local bifurcations around it in terms of k and x. It is worth noting that, since α appears in the definition of x, the analysis will be set in terms of the original parameters of the model, by fixing α. This investigation will then allow us to define the dynamic properties of the model in terms of marginal costs' ratio and the speed of adjustment, for a fixed level of α.

Remark 7 *The case in which both denominators in (3.20) and (3.21) are equal to zero simplifies the analysis and such occurrence will be discussed at the end of the section.*

Remark 8 *The expressions in (3.20) and (3.21) do not allow to obtain a functional relation binding α to k and the marginal costs' ratio. Because of the crucial role of differentiation, we will discuss through numerical analysis how the parameter α is decisive in defining the dynamics of the model.*

[7] We note that, because of the homogeneity of degree -2 of functions F_l, the relations $F_l(\alpha, c_1, c_2) = \frac{f_l(\alpha, x)}{c_2^2}$ with $l = 1, 2$ holds.

3.4.1 *Shapes of graphs of* f_1 *and* f_2

In order to describe the behaviour of the graphs of the functions defined in (3.20) and (3.21), we can first notice that (i) $\lim_{x\to 0} f_1(\alpha,x) = \lim_{x\to 0} f_2(\alpha,x) = 0$, that is both curves approach the origin of the axes in the plane $(x,\widetilde{k})$ and (ii) $\lim_{x\to+\infty} f_l(\alpha,x) = 0$ with $l = 1,2$, that is the x-axis represents a horizontal asymptote for both curves. By a direct inspection of $h(\alpha,x)$ and $z(\alpha,x)$, we get the following Lemma:

Lemma 3 *Let $h(\alpha,x)$, $z(\alpha,x)$, $f_1(\alpha,x)$ and $f_2(\alpha,x)$ defined in (3.18), (3.19), (3.20) and (3.21), respectively.*
(a) If $\alpha^2 + \frac{1}{4}\alpha - 1 > 0$, then there exists a unique $\overline{x}_1 \in (0,+\infty)$ such that $h = 0$. Therefore, f_1 is not defined in $x = \overline{x}_1$. Otherwise, f_1 is defined for every x in $(0,+\infty)$;
(b) if $\alpha^2 + \frac{1}{2}\alpha - 1 > 0$, then there exists a unique $\overline{x}_2 \in (0,+\infty)$ such that $z = 0$. Therefore, f_2 is not defined in $x = \overline{x}_2$. Otherwise, f_2 is defined for every x in $(0,+\infty)$.

Proof.
(a) Consider the function $h(\alpha,x)$, defined in (3.18). We have that $\alpha^2 - \frac{5}{4}\alpha + 2 > 0$. Being the Δ of $h(\alpha,x)$ always positive, we can notice that the potential changes of sign for $h(\alpha,x)$ depend on the sign assumed by $\alpha^2 + \frac{1}{4}\alpha - 1$. In particular, for $\alpha^2 + \frac{1}{4}\alpha - 1 > 0$, we have that $h(\alpha,x)$ changes its sign at most one time and then there exists a unique positive value $\overline{x}_1$ such that $h(\alpha,\overline{x}_1) = 0$; for $\alpha^2 + \frac{1}{4}\alpha - 1 = 0$, $h(\alpha,x)$ becomes a polynomial of degree 1 w.r.t. x and it assumes only negative values; for $\alpha^2 + \frac{1}{4}\alpha - 1 < 0$, $h(\alpha,x)$ is the sum of three negative terms, then it assumes always negative values. Finally, the result follows.
(b) Analogously, consider the function $z(\alpha,x)$, defined in (3.19). We have that $\alpha^2 - \frac{3}{2}\alpha + 2 > 0$. Being the Δ of $z(\alpha,x)$ always positive, we can notice that changes of sign in $z(\alpha,x)$ depend on the sign assumed by $\alpha^2 + \frac{1}{2}\alpha - 1$. In particular, for $\alpha^2 + \frac{1}{2}\alpha - 1 > 0$, the sign of $z(\alpha,x)$ changes at most one time and there exists a unique positive $\overline{x}_2$ such that $z(\alpha,\overline{x}_2) = 0$; for $\alpha^2 + \frac{1}{2}\alpha - 1 = 0$, $z(\alpha,x)$ becomes a polynomial of degree 1 w.r.t. x and it assumes only negative values; for $\alpha^2 + \frac{1}{2}\alpha - 1 < 0$, $z(\alpha,x)$ is the sum of three negative terms and then it assumes only negative values. Therefore, the result follows. ■

On the basis of the previous Lemma, we can state the following Proposition:

Proposition 20 *Let $\alpha_1 = \frac{1}{8}(\sqrt{65} - 1)$ and $\alpha_2 = \frac{1}{4}(\sqrt{17} - 1)$. Then,*
(a) For $\alpha \in (0,\alpha_2)$, $\overline{x}_1$ and $\overline{x}_2$ do not exist;
(b) for $\alpha \in (\alpha_2,\alpha_1)$, $\overline{x}_2$ exists while $\overline{x}_1$ does not exist;
(c) for $\alpha \in (\alpha_1,1)$, $\overline{x}_1$ and $\overline{x}_2$ exist.

Proof. By solving $\alpha^2 + \frac{1}{4}\alpha - 1 = 0$ and $\alpha^2 + \frac{1}{2}\alpha - 1 = 0$, we obtain the values of α_1 and α_2,respectively. Therefore, the relation $\alpha_2 < \alpha_1$ is straightforward.

(a) From Lemma 3, we can deduce that, in the interval $(0, \alpha_2)$, $\overline{x}_2$ does not exist and then the inequality $\alpha_2 < \alpha_1$ guarantees that also $\overline{x}_1$ does not exist. (b) The same inequality implies that there exists a range (α_2, α_1) in which only $\overline{x}_2$ exists and then the asymptote for f_2 is the unique admitted. (c) Finally, at the right of α_1 both $\overline{x}_1$ and $\overline{x}_2$ exist. ■

Remark 9 *In the previous Proposition we analyse the conditions for which $\overline{x}_1$ and $\overline{x}_2$ exist. We notice that both values are exclusively dependent on α ($\overline{x}_1 = \overline{x}_1(\alpha), \overline{x}_2 = \overline{x}_2(\alpha)$). Regarding the original parameters of the model, given α, we have that positive values of c_1 and c_2 exist such that*

$$\left(\frac{c_1}{c_2}\right)^{\alpha} = \overline{x}_i, \quad i = 1, 2$$

holds. In particular, being $\alpha \in (0, 1]$, if $\overline{x}_i > 1$ we have $\frac{c_1}{c_2} > 1$ while if $\overline{x}_i < 1$ we have $\frac{c_1}{c_2} < 1$.

In order to have a graphical insight into what is shown above, we refer the reader to Figure 3.1.

3.4.2 *Intersections between graphs of f_1 and f_2*

The existence of intersection points between f_1 and f_2 can be analysed in the plane $(x, \widetilde{k})$, as α varies. As far as this is concerned, the following proposition holds:

Proposition 21 *There exists a unique intersection point $(x^*, \widetilde{k}^*) = \left(\frac{4}{5\alpha-4}, \frac{8(5\alpha-4)}{25\alpha}\right)$ between the curves $\widetilde{k} = f_1$ and $\widetilde{k} = f_2$ in the plane $(x, \widetilde{k})$. It is feasible, that is $x^*, \widetilde{k}^* > 0$, if and only if $\alpha > \alpha^* = \frac{4}{5}$.*

Proof. Solving the equation

$$f_1(\alpha, x) = f_2(\alpha, x) \tag{3.22}$$

in terms of x, we have that a unique solution $x^* = \frac{4}{5\alpha-4}$.
Therefore, $x^* \in (0, +\infty)$ if and only if $\alpha > \frac{4}{5}$. By evaluating f_1 (or f_2) at x^*, the positive value $\widetilde{k}^* = \frac{8(5\alpha-4)}{25\alpha}$ is derived. ■

Corollary 2 *The positive intersection point $(x^*, \widetilde{k}^*) = \left(\frac{4}{5\alpha-4}, \frac{8(5\alpha-4)}{25\alpha}\right)$ exists if and only if $\overline{x}_2$ exists, where $\overline{x}_2$ is defined in Lemma 3.*

Proof. Recalling the definition of $\overline{x}_2$ in Lemma 3 and that $\alpha_2 = \frac{1}{4}(\sqrt{17} - 1)$, the proof is straightforward because $\alpha_2 < \alpha^*$. ■

Remark 10 *The equation*

$$\left(\frac{c_1}{c_2}\right)^{\alpha} = \frac{4}{5\alpha - 4}$$

has a solution in terms of α *in the interval* $(\frac{4}{5}, 1]$ *only if* $\frac{c_1}{c_2} > 1$*. The equation has no solutions when* $\frac{c_1}{c_2} < 1$*.*

Remark 11 *The previous Propositions allow to deduce that (i) in the interval* $(0, \alpha_2)$*,* f_1 *assumes only positive values while* f_2 *assumes only negative ones and (ii) in the interval* (α_2, α^*)*, both* f_1 *and* f_2 *assumes only positive values but* f_2 *assumes higher values that* f_1*.*

3.4.3 Local stability of the Nash Equilibrium

In the light of the results discussed above, we can formulate the following Proposition on the local stability of E^*:

Proposition 22 *(a) If (i)* $\alpha \in (0, \alpha^*)$*, then* E^* *is locally asymptotically stable for* $\widetilde{k} < f_1(\alpha, x)$*; (ii) for* $\alpha = \alpha^*$*,* E^* *loses its stability through a Flip bifurcation; (iii) otherwise,* E^* *is unstable.*
(b) If (i) $(0, x^*)$ *the following cases arise: for* $\alpha = \alpha^*$*,* E^* *loses its stability through a Flip bifurcation while, for* $\alpha \in (\alpha^*, 1)$*,* E^* *is locally asymptotically stable for* $\widetilde{k} < f_1(\alpha, x)$*. (ii) If* $(x^*, +\infty)$*, the following cases arise: for* $\alpha = \alpha^*$*,* E^* *loses its stability through a Neimark-Sacker bifurcation while, for* $\alpha \in (\alpha^*, 1)$*,* E^* *is locally asymptotically stable for* $\widetilde{k} < f_2(\alpha, x)$*. (iii) Otherwise,* E^* *is unstable.*

Proof. (a) By recalling Remark 11, in the interval $(0, \alpha^*)$ we have that E^* is stable for every $\widetilde{k} < f_1(\alpha, x)$. On the contrary, for every $\widetilde{k} > f_1(\alpha, x)$ E^* loses its stability due to a Flip bifurcation, generated in the geometric place of the points $(x, \widetilde{k})$ such that $\widetilde{k} = f_1(\alpha, x)$.
(b) In the interval $(\alpha^*, 1)$, the positive intersection point x^* exists. By considering that $\lim_{x \to 0} f_l(\alpha, x) = \lim_{x \to +\infty} f_l(\alpha, x) = 0$ with $l = 1, 2$, we have that E^* is stable for $\widetilde{k} < f_1$ in the interval $(0, x^*)$ and for $\widetilde{k} < f_2$ in the interval $(x^*, +\infty)$. On the contrary, the couples $(x, \widetilde{k})$ such that $\widetilde{k} = f_1$ define the geometric place of points in which the fixed point is destabilised through a Flip bifurcation in the interval $(0, x^*)$, while the couples $(x, \widetilde{k})$ such that $\widetilde{k} = f_2$ the geometric place of points in which the fixed point is destabilised through a Neimark-Sacker bifurcation in the interval $(x^*, +\infty)$. ■

For the sake of completeness, the following proposition discusses the stability of E^* in such cases where $h(\alpha, x)$ or $z(\alpha, x)$ vanish:

Proposition 23 *(a) If* $h(\alpha, x) = 0$ *and* $\alpha \in (\alpha_1, 1)$*, then* E^* *may lose its stability via Neimark-Sacker bifurcation;*
(b) if $z(\alpha, x) = 0$ *and* $\alpha \in (\alpha_2, 1)$*, then* E^* *may lose its stability via Flip bifurcation.*

Proof. (a) $h(\alpha,x)$ is equal to zero if and only if $x=x_1^*$ or $x=x_2^*$ where

$$x_{1,2}^* = \frac{1}{2}\,\frac{4\,\alpha^2 - 5\,\alpha + 8 \pm \sqrt{137\,\alpha^2 - 104\,\alpha^3 + 16\,\alpha^4}}{4\,\alpha^2 + \alpha - 4}.$$

We have that $x_1^* > 0$ for $\alpha \in (\alpha_1, 1)$ while $x_2^* < 0$ for every α. By substituting x_1^* in the second condition of (3.14), we have that the inequality $1 + Tr(J_{E^*}) + Det(J_{E^*}) > 0$ is fulfilled by every $\alpha \in (\alpha_1, 1)$. This implies that, in the interval $(\alpha_1, 1)$, the Jury conditions are satisfied if and only if the condition $1 - Det(J_{E^*}) > 0$ is satisfied; (b) $z(\alpha, x)$ is equal to zero if and only if $x = x_3^*$ or$x = x_4^*$, where

$$x_{3,4}^* = \frac{1}{2}\,\frac{2\,\alpha^2 - 3\,\alpha + 4 \pm \sqrt{33\,\alpha^2 - 28\,\alpha^3 + 4\,\alpha^4}}{2\,\alpha^2 + \alpha - 2}.$$

We have that $x_3^* > 0$ for $\alpha \in (\alpha_2, 1)$ while $x_4^* < 0$ for every α. By substituting x_3^* in the third condition of (3.14), we have that the inequality $1 - Det(J_{E^*}) > 0$ is fulfilled by every $\alpha \in (\alpha_2, 1)$. This implies that, in the interval $(\alpha_2, 1)$, the Jury conditions are satisfied if and only if the condition $1 + Tr(J_{E^*}) + Det(J_{E^*}) > 0$ is satisfied. ■

Remark 12 *Proposition 22 allows to deduce that a destabilisation via Neimark-Sacker with respect to the parameter $\widetilde{k}$ may occur only for really high values of α ($\alpha \in (\alpha^*, 1)$). In addition, by combining Propositions 21 and 22, it must hold:*

$$\frac{c_1}{c_2} > \left(\frac{4}{5\alpha - 4}\right)^{\frac{1}{\alpha}}. \tag{3.23}$$

The inequality in (3.23) implies that, for $\alpha \in (\alpha^*, 1)$, E^* may be destabilised via Neimark-Sacker only if c_1 is at least the quadruple of c_2.

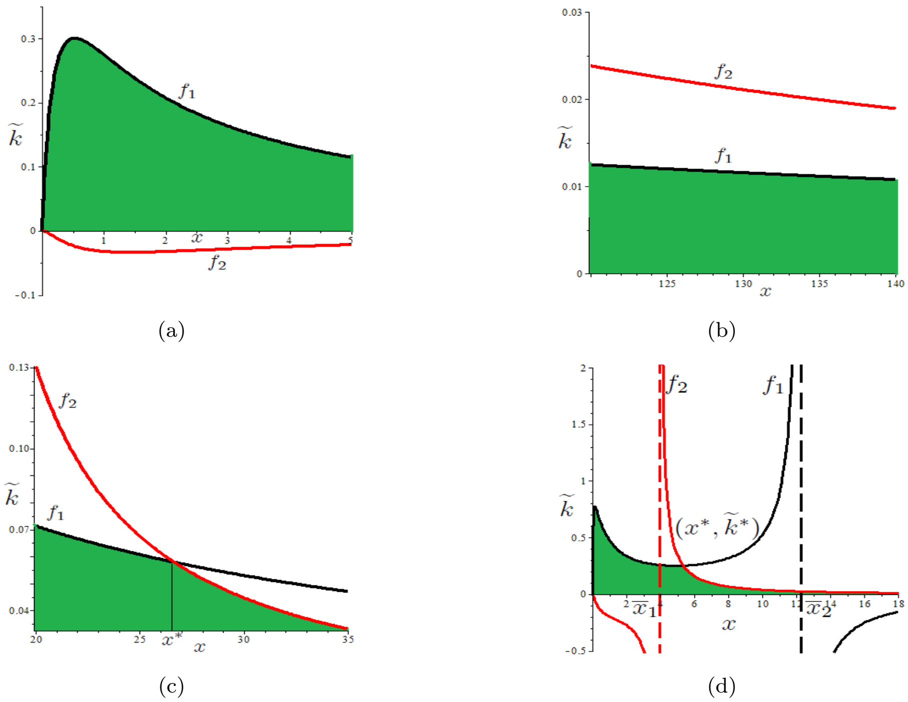


Figure 3.1: Different stability regions (areas in green) of E^* in the plane $(x, \widetilde{k})$, defined by the bifurcation curves f_1 (depicted in black) and f_2 (depicted in red). (a) The fixed point may lose its stability only through a Flip bifurcation, $\alpha = 0.55$; (b) both the bifurcations curve are in the positive plane, but E^* may destabilise itself only through a Flip bifurcation, $\alpha = 0.798$; (c) the stability region when both the bifurcation curves are in the positive plane and intersect each other in x^*, $\alpha = 0.83$; (d) the stability region when there exist both the intersection point $(x^*, \widetilde{k}^*)$ and the asymptotes $\overline{x}_1$ and $\overline{x}_2$, $\alpha = 0.95$.

The graphs in Figure 3.1 represent a numerical confirmation of the results shown in Propositions 20, 21 and 22. In particular, Panels (a) and (b) of Figure 3.1 numerically confirm the case (a) in Proposition 22 while Panel (c) of Figure 3.1 shows a numerical example of case (b) in Proposition 22. Finally, Panel (d) in Figure 3.1 describes the stability region for a value of α such that (i) f_1 and f_2 are not defined at a point ($\overline{x}_1$ and $\overline{x}_2$, respectively), as stated in Proposition 20 and (ii) f_1 and f_2 have an intersection point $(x^*, \widetilde{k}^*)$, as stated in Proposition 21. Moreover, with regard to the results shown in Proposition 23, Panel (d) in Figure 3.1 allows to notice that, at $\overline{x}_1$ and $\overline{x}_2$, the Nash equilibrium is stable if the configuration of the parameters defines a point in the region depicted in green.[8]
Considering the results of the local analysis, in what follows we will discuss some

[8]The configurations in Panels (c) and (d) of Figure 3.1 can be obtained only when $c1 > c2$.

dynamic scenarios.

3.4.4 *Bifurcations and stability*

The stability conditions provided in Proposition 22 allow us to deduce relevant information on the effect of both the speed of adjustment and the marginal costs ratio. In particular, starting from a parameter configuration for which the equilibrium is stable, an increase of k, leaving all the other parameters as fixed, imply a destabilisation of E^* through a Flip or Neimark-Sacker bifurcation, in line with the results of Agliari et al. (2016) and the majority of literature. The hump-shaped behaviour of the graph of f_1 induces a twofold role for x. Indeed, as suggested by the Panel (a) in Figure 3.1, we observe that for a fixed value of the speed of adjustment, the Nash equilibrium is first unstable, then stable and finally unstable again.

The bifurcation diagrams in Figure 3.2, performed with respect to the speed of adjustment k, numerically confirm the theoretical results proved in Proposition 22. In Panel (a) of Figure 3.2, we can notice that the Nash equilibrium is locally stable for low values of k and it undergoes a Flip bifurcation at $k = k_{Flip} \simeq 0.18859649$ generating a stable 2-period cycle. As the speed of adjustment further increases, a sequence of period doubling bifurcations generates cycles of a higher period leading to chaos. Differently, Panel (c) of Figure 3.2 shows an example in which, as k varies, a Neimark-Sacker bifurcation takes place for $k = k_{ns} \simeq 0.036752$. Finally, Panel (d) of Figure 3.2 describes a numerical example of how instability and complex phenomena may occur regardless of marginal costs ratio. Indeed, the graph shows that, for both $c_1 < c_2$ and $c_1 > c_2$, chaotic regimes may arise. From an economic point of view, we can then observe that scenarios of instability may occur both if the largest impact on the market is held by the firm adopting the LMA (case $c_1 < c_2$) and if the largest impact is held by the firm adopting the gradient-like mechanism (case $c_1 > c_2$).

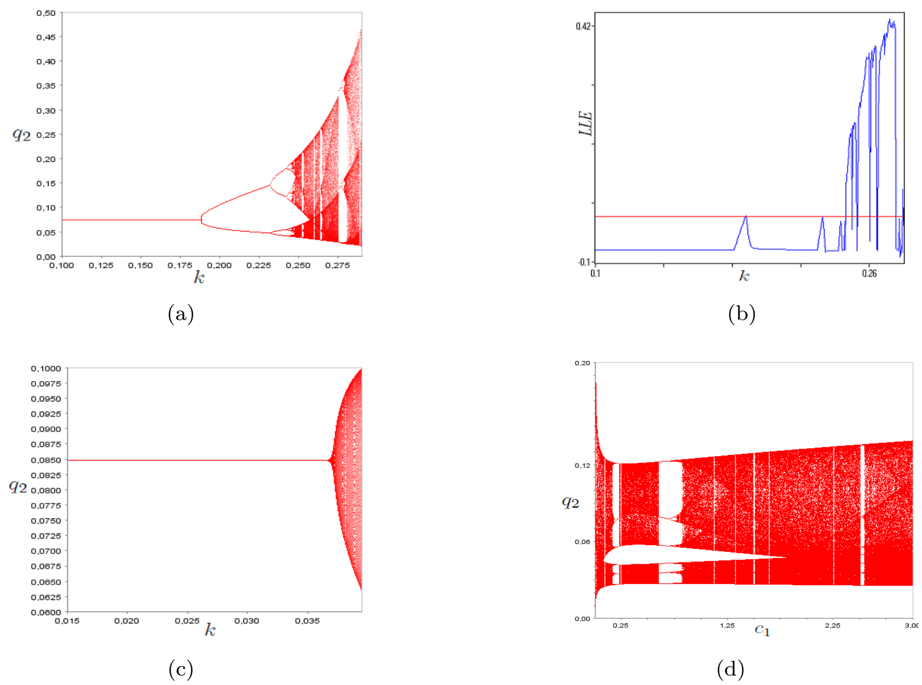


Figure 3.2: (i) Bifurcation diagrams of strategy q_2 as k varies: (a) Parameter set: $\alpha = 0.4, c_1 = 0.064, c_2 = 1$. The Nash equilibrium loses its stability through a Flip bifurcation; (b) Largest Lyapunov exponent with respect to k associated to Panel (a); (c) Parameter set: $\alpha = 0.97, c_1 = 10, c_2 = 1$. E^* undergoes a Neimark-Sacker bifurcation. (d) Parameter set: $\alpha = 0.19, c_2 = 1, k = 0.13$. Bifurcation diagram of strategy q_2 as c_1 varies in the interval $[0.002, 3]$, where E^* is destabilised in both cases $c_1 < c_2$ and $c_1 > c_2$.

The following bifurcation diagrams emphasise the peculiar role played by the differentiation parameter α.[9] In particular, we can observe that the adoption of different parameter sets affects the manner in which α alternatively destabilises and stabilises the fixed point. Indeed, the following scenarios arise: in Panel (a) of Figure 3.3, assuming that $c_1 > c_2$, an increase in the product differentiation (that is, a decrease in the value of α) destabilises the Nash equilibrium via a flip bifurcation and a stable 2-cycle appears around the unstable E^*. Then, as α reduces, a sequence of flip bifurcations occurs leading to chaos. Panel (b) of Figure 3.3 confirms the result in (a) by assuming $c_1 < c_2$. In Panel (c) of Figure 3.3, the phenomenon described in (a) is only partially confirmed. Indeed, although a decrease in α to low values implies the same result of (a), we can notice that for high values of α a further destabilisation occurs via flip bifurcation and as α tends to 1 another cascade of flip bifurcations leading to chaos appears.

[9]In dynamic exercises, we have considered values of α values such as to avoid negativity problems.

Panel (d) of Figure 3.3, in line with (c), shows the dual role of α but highlights the occurrence of a destabilisation, for high values of α, via Neimark-Sacker bifurcation. These examples reveal that the degree of differentiation may have an ambiguous role and as a consequence, given an appropriate parameter set, both low and high values of α may induce instability. In particular, the numerical exercises show that (i) α is destabilising when it assumes low values, regardless of the marginal costs' ratio (as in Agliari et al., 2016) and (ii) how, differently from Agliari et al. (2016), the fixed point may be destabilised also for high levels of α (that is, when goods are increasingly perceived as indistinguishable) via Flip bifurcation or via Neimark-Sacker bifurcation, if c_1 is sufficiently larger than c_2 (see Remark 12). From an economic point of view, these results induce interesting conclusions. The result (i) suggests that for $\alpha \to 0$ goods are basically independent and in fact firms operate in distinct markets, characterised by isoelastic demands with elasticity close to 1, where they are considered as monopolists. In such a case, prices react little to changes in the amount of goods placed on the market and are not able to bring the market back to a stationary equilibrium.[10] Instead, The result (ii) suggests that, for a low degree of product differentiation, the goods start to be perceived as indistinguishable and competition is high. In such a case, prices excessively react to changes in the amount of goods and are not able to bring the market back to a stationary equilibrium. In the case of $\alpha \to 1$, we can note also that instability scenarios may occur even when the largest impact on the market is held by the firm who adopt the LMA (the case $c_1 > c_2$), which is usually stabilising. The latter represents a really counterintuitive result both compared with those presented in Agliari et al. (2016), where homogeneous gradient-like decisional mechanisms are considered, and the possible scenario in which the market is composed of two firm adopting LMA, for whom there is stability as α increases. Therefore, we can conclude that the instability is due to the interaction between heterogeneous decisional mechanisms, namely the gradient-like mechanism and LMA.

[10] In this context, the marginal profit that drives the decision-making mechanism is such as to induce strong fluctuations in the decisions of firm 2. Indeed, in order to maintain the nonnegativity of the produced quantities, Equation (3.11) (and therefore also the first equation in the map M), should be rewritten as $q_{2,t+1} = max\left(0, q_{2,t} + k\frac{\partial \pi_2(q_{1,t}, q_{2,t})}{\partial q_{2,t}}\right)$.

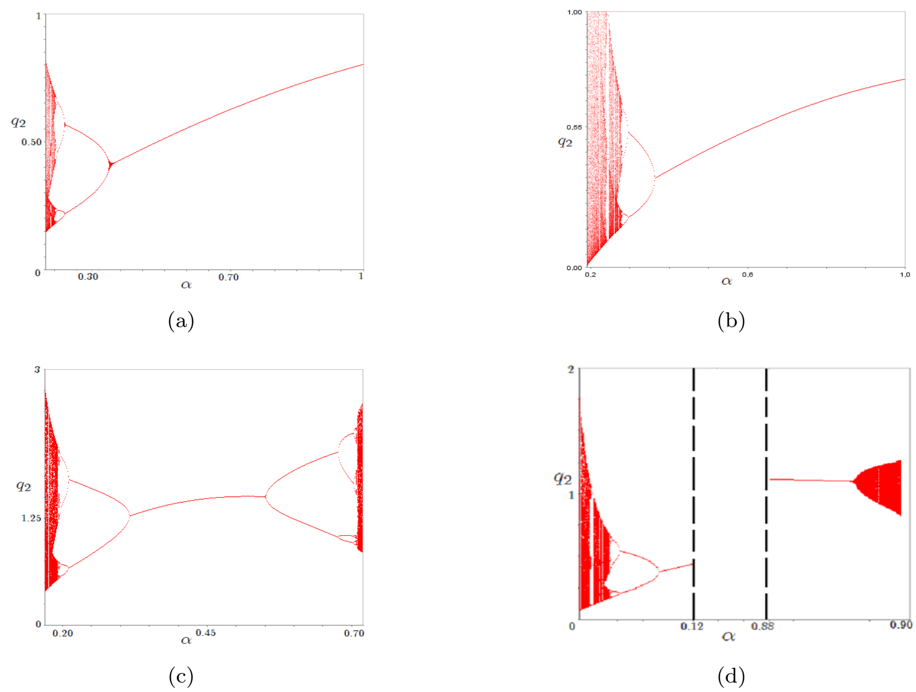


Figure 3.3: (a) Parameter set: $c_1 = 0.65, c_2 = 0.25, k = 3$. A cascade of period-doubling bifurcations occurs as α decreases, in the case $c_1 > c_2$; (b) Parameter set: $c_1 = 0.08, c_2 = 0.25, k = 3$. A cascade of period-doubling bifurcations occurs as α decreases, in the case $c_1 < c_2$; (c) Parameter set: $c_1 = 0.97, c_2 = 0.05, k = 45$. From the left to the right, first a sequence of period-halving bifurcations occurs, then a phase of stability arises for $\alpha \in (0.3167, 0.5510)$ and at the end a sequence of period-doubling bifurcation appears; (d) Parameter set: $c_1 = 0.97, c_2 = 0.05, k = 15$. From the left to the right, first a sequence of period-halving bifurcations occurs, then a phase of stability arises for $\alpha \in (0.0775, 0.8969)$ and at the end a Neimark-Sacker bifurcation appears.

Remark 13 *Numerical experiments performed above suggest that the Flip bifurcation has always a supercritical nature. Differently, The Neimark-Sacker bifurcation may experience a switch from a supercritical nature to a subcritical one.*

Figure 3.4 furnishes another interesting numerical example. More in depth, the graph shows that starting from the initial condition $(q_1^0, q_2^0) = (0.0016, 0.1)$ and varying the speed of adjustment k, a Neimark-Sacker bifurcation takes place at $k = \hat{k}_{ns} \simeq 0.02924342$. Passing the critical value $\hat{k}_{ns}$, a quasi-period behaviour starts and lasts until $k \simeq 0.03254934$ from which such regime is replaced by a sequence of frequency-locking intervals. In these intervals, the motion along the stable closed invariant curve becomes captured by a periodic cycle therein contained. As the graph suggests, for values of k sufficiently high the system falls in a chaotic regime.

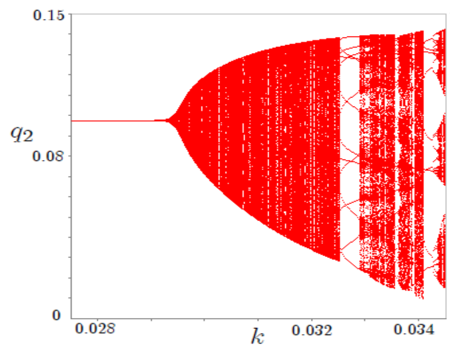


Figure 3.4: Bifurcation diagram with respect to k for $\alpha = 0.985$, $c_1 = 10$, and $c_2 = 1$.

Regarding the role of k, in Figure 3.5 we provide some phase plane diagrams, which show (i) the initial quasi-periodic dynamics with an attracting invariant closed curve (see Panels (a) and (b)), (ii) the successive period-46 cycle generated by one of the frequency-locking intervals (see Panel (c)) and (iii) the unconnected cyclical areas after the frequency-locking (Panels (d)). As the speed of adjustment further increases, (iv) a 9-pieces chaotic attractor appears (see Panel (e)). For $k \simeq 0.03449750$, a final bifurcation occurs and almost all trajectories become unfeasible.

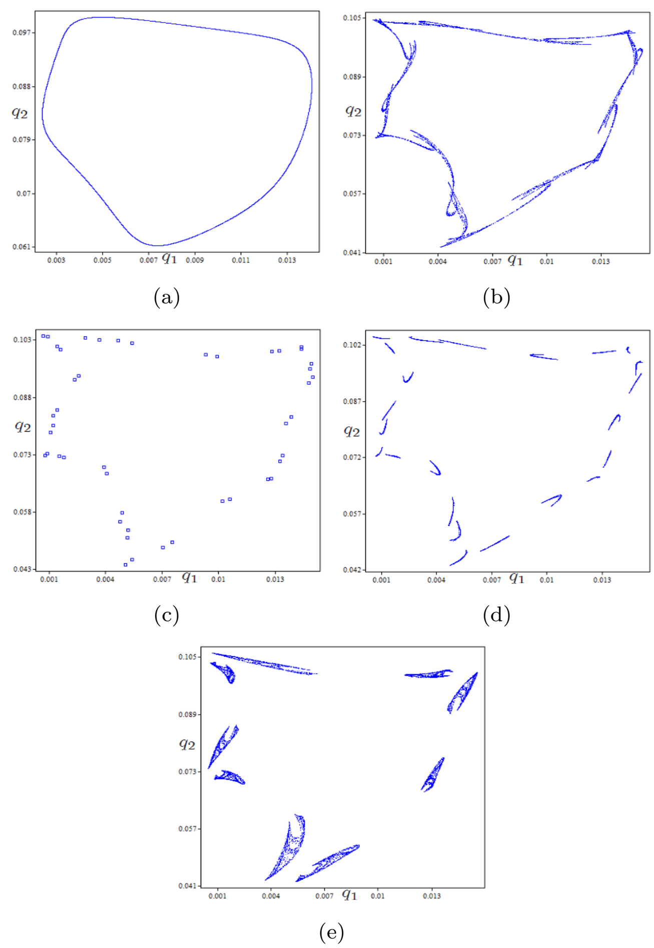


Figure 3.5: Parameter set: $\alpha = 0.985, c_1 = 10, c_2 = 1$. From left to right, top to bottom. Phase plane diagrams for different values of the parameter k. (a) $k = 0.0303597$; (b) $k = 0.0335548$; (c) $k = 0.0336313$ (d) $k = 0.0336568$; (e) $k = 0.03449750$.

Finally, Figure 3.6 shows that starting from two different initial conditions for q_1 (given the same initial value for q_2), after the transient phase, dynamics settle down to different periodic cycles. Then, the coexistence of two attractors appears.

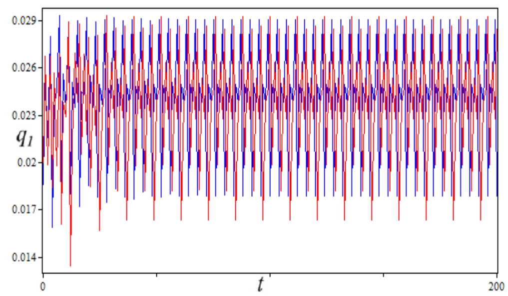


Figure 3.6: Parameter set: $\alpha = 0.88, c_1 = 5, c_2 = 1, k = 0.22857$. The stable 4-cycle, represented by the time series depicted in blue ($q_1(0) = 0.018606$), coexists with the stable 12-cycle described by the time series depicted in red ($q_1(0) = 0.02$).

3.5 Conclusions

In this chapter, we analysed the dynamics of a Cournot duopoly with differentiated goods and boundedly rational firms adopting heterogeneous decisional mechanisms to adjust the quantity of output produced. We showed that the differentiation parameter has an ambiguous role because both high and low levels of product differentiation may destabilise the Nash equilibrium, leading to cyclical behaviours and chaotic dynamics as well. This is a really counterintuitive result both compared with those presented in Agliari et al. (2016), where both firms adopt a gradient-like decisional mechanism, and the possible scenario in which the market is composed of two firms adopting LMA, for whom stability persists as the degree of differentiation varies. Therefore, we can conclude that the main element generating instability is specifically the interaction between heterogeneous decisional mechanisms, namely the gradient-like mechanism and LMA. With regard to this destabilising role of product differentiation, we also provided different parameter configurations for which the equilibrium loses its stability through Flip and/or Neimark-Sacker bifurcations. In addition, we have described, for appropriate ranges of parameters, the occurrence chaotic dynamics and coexistence of attractors.

Chapter 4

The influence of social norms and the effects of intrinsic costs on the labour force participation of women

4.1 Introduction

The role that women have in society and, in particular, the extent of their contribution to market activities (e.g. work) and non-market activities (e.g. family) continue to occupy a central role in public opinion and in decades-long debates among social scientists, with some professions more under scrutiny than others.[1] Female labour participation and, in particular, gender gap (See O'Neill, 2003) are ultimately a very widespread issue (https://www.ft.com/gender-pay-gap). Economists have attempted to contribute to the debate. Some contributions in the field have considered gender issues mostly as a special case, an extension to the standard framework, allowing for example a generic individual (or a household) to allocate time not just between work and leisure, but among work, leisure and family. Other contributions, recognising that female labour may have different characteristics and comparative dis/advantages compared to male supply, explain female labour participation as the result of technological structural change. When societies move from manufacturing to service economies the demand for female labour supply may increase and gender gaps worldwide can be interpreted as the result of different elasticities of substitution among male

[1] Recently the magnifying glass has been pointed to the economics professions. See for examples Hengel (2017), Wu (2017) and related debates.

and female labour input (see Olivetti and Petrongolo, 2014). Other contributions argue that, because of family commitments, female labour may be seen by employers as riskier compared to male labour, even if men and women may be equally productive. It follows that the way household tasks are allocated in the family affects the way the input of a woman is assessed by employers. Similarly, the fact that a household may decide to let the man be the first earner in the family is also the result of the fact that the labour market internalises unequal involvement in family matters between man and woman and therefore the gender pay gap. In other words, inequality in the family produces inequality in the market place and viceversa. Similar forces are at play in those contributions that consider human capital accumulation and life cycle models (see Attanasio et al., 2008; Park, 2018). The approach of the contributions considered so far is based on a unitary perspective of the household, where women are assumed to have a comparative advantage in taking care of home matters and a family decides rationally how to allocate the time of men and women. In this sense, gender inequality is the result of rational economic decisions, similar to the key messages provided by international trade theory.

Other contributions have moved away from this unitary model of the household to consider more in detail the decision dynamics in the family. Contributions have considered forms of non-cooperative bargaining, cooperative bargaining,[2] and a collective approach (Chiappori, 1992). These contributions highlight the fact that women have individual preferences and consider possible sources of inequality (possibly also created by social norms) in the bargaining.[3]

To sum up, in general economic models assume that all women have (and want) a family and their contribution to society, both in the household and in the workplace, is a result on their interaction with men. Specifically women contribution is the result of bargaining and specialisation with men in the household (where it is often the case in which are assumed to have a comparative advantage in caring for children and housework) and competition with men in the workplace. This approach also implicitly assumes that women are essentially a rather homogenous group and that, if given proper support by society (including employers and partners) they would be naturally inclined to have a family and satisfy an innate need for motherhood. The approach looks at women as fully altruistic, rational economic agents who do not experience utility outside of the household. This is obviously not necessarily true and does not come without consequences. Assuming that women form a homogenous group implies that policy intervention can be targeted at all women without side effects or unintended consequences. Also, given that the actions of women tend to be seen as a product of social constraints, mostly created by male behaviour, another implication is that much of the policy intervention should directed at employers and male behaviour (for example ensuring equal opportunities in the workplace

[2]In this regard, see McElroy and Horney (1981), Lundberg and Pollak (2003) and Basu (2006).

[3]Identity theory explains that women's preferences and decisions may be affected by social costs/rewards for conforming to a behaviour expected by society. See Akerlof and Kranton (2000).

and at home). This approach unavoidably tends to suffer from a *male bias* and a patriarchal way to look at society in general, where all individuals are assumed to aspire to the highest positions in society, and at women in particular, where women are assumed to be a homogenous group with identical preferences and characteristics. Moreover, the theoretical framework of economic models that have studied the decision of women to participate in the labour market has been in general based on the assumption that economic agents, men and women, are rational. The decision to contribute to work in the household and the market has been framed as the result of a rational bargaining process in the family or as the rationally far sighted investment decision in human capital of men and women.

While the behaviour and attitude of men in the workplace and at home undoubtedly can play an important role in explaining some of the evidence observed by social scientists regarding the role of women in society and the workplace, assuming that the behaviour of women is defined by what the society (and especially men) allow them to do is rather limited, at least when discussing societies in developed countries, and potentially harmful if assumed when devising policy.

In sociology, the framework of Preference Theory (PT) offers an alternative way to look at the problem, at least in developed countries. PT assumes that women (differently than men) are a rather heterogeneous group. Specifically, the theory is based on four key principles derived from empirical observation: society, at least in most developed countries, has changed to provide women with more opportunities; women are not a homogenous group, but have heterogeneous preferences and priorities; heterogeneity of preferences among women potentially may create conflicting interests; because of their heterogenous preferences, women may react differently to the same policy. PT classifies women in three groups:[4] *work centred*, *family centred* and *adaptive*. Women who belong to the first group prioritise work and career; they choose to postpone (and even avoid) motherhood, not because of financial necessity, but because of their preference toward a professional career. Women who belong to the second group, instead, prioritise family and motherhood and do not consider contributing to the work market; they do not access the labour market, not because their financial conditions necessarily allow them to do so, but because they see a benefit in raising children and not in spending time in the workplace. Of course, in the general the majority of women tend to belong to the last group. Adaptive women are those whose preferences include both work and motherhood and they need to choose how to allocate their time between these two activities.

An important implication of PT is that, since women may have heterogeneous preferences and react differently to policies, it becomes essential for policymakers to develop a clear understanding of the preferences of different groups of women so that gender policies can be carefully targeted and assessed. In particular, when studying female labour participation and gender gaps, it is often important to take a dynamic perspective. If women have specific preferences

[4]This is clearly a simplification. The framework of PT, however, provides a coherent way to look at women's behaviour in a way rarely adopted by social scientists.

and these preferences may be influenced by social norms and culture, it follows that long run labour participation and gender gaps can be the result of cultural change and evolution of social norms (see Bowles, 1998; Bisin and Verdier, 2000). Contributions in the literature that consider cultural transmission and the long run choices of women assume that women are married, with children and concerned about the costs of pursuing a professional career. Such costs may be linked to the way society sees employed women or could be related to the fact that women may internalise the effect that time spent at work may have on the education of their children. Socially interacting with women employed in previous generations, women can learn of these costs, update their believes and intertemporally optimise their time allocation, allowing cultural transmission and evolution of social norms (Fernández et al., 2004; Fogli and Veldkamp, 2011; Fernández, 2013). These contributions, therefore, still tend to frame women to the role of wife/mother who can (similar to the human capital/life cycle literature mentioned above) rationally take complex decisions.

In this paper we want to move away from the idea that all women have the innate desire to be wives and mothers, that their utility includes time spent at work only as a (financial and social) costs and that female labour participation and gender gaps are the result of rational economic choices, often constrained by patriarchal societies and families. Adopting the lenses of PT, this paper provides an economic model that studies the way women's role in society may evolve; specifically we shall focus on the group of adaptive women and see how their allocation of effort between family and work is affected by payment schemes in the workplace, various *intrinsic costs* they have depending on their inclinations and, importantly, by social interaction with other women, who may have different innate preferences. We are indeed expanding the PT framework and assuming that there may be heterogeneity among adaptive women too. Some of them will be more naturally inclined to contribute in the labour market while other will find more rewarding to contribute in the household. We are deliberately leaving men out of the picture. This is clearly a simplification, but it allows us to stress the fact that many decisions of women are the result of voluntary actions based on heterogeneous preference and not necessarily the result of behaviour imposed by men. We shall however allow social norms to play a role in our model in which they will influence (but not determine) the long run decisions of women. In other words, adopting some economics jargon, social norms will be an important element of the utility function of women, but not a binding constraint that will define their actions.

Considering an evolutionary game (Weibull, 1995) allows us to study how different types (in terms of their inclination toward work or family) of women decide to allocate their effort in the short run and how women can revise their preferences ("evolve") in time. Our evolutionary game approach introduces a realistic degree of bounded rationality in the decision of women on how and how much to contribute to society. Their behaviour is essentially imitative and the spreading of a type of inclination (career orientation and family orientation) in the population of women is ultimately defined by social interaction and learning. Specifically, in line with *word of mouth dynamics* (see Dawid, 1999),

we assume that women socially interact and, from these interactions, they can learn about the utility of other women, in particular those who with different preferences. Social interaction, therefore, plays an important role in allowing women over time to learn and change inclination, evolving their preferences in the long run. Our findings show that both scenarios in which the population converges to monomorphic configurations and scenarios where the population converges to a polymorphic composition (that is, a stable inner equilibrium for the system may exist) are achievable. Moreover, due to the willingness to switch inclination and the interplays between social norms and intrinsic costs may lead to the emergence of periodic cycles as well as chaotic regimes.
The rest of the chapter is organised as follows. In Section 2 we describe the mathematical model and the static analysis; in Section 3 we depeen the dynamics generated by the model and the particular role played by crucial parameters; Section 4 concludes.

4.2 The Model

Let us consider a population of women who are not a priori deciding to be childless (i.e. those called *work-centred* by PT) or voluntarily unemployed (i.e. those called *home-centred* by PT). Instead, we are considering a group of women (PT would call them *adaptive*) who are employed in the same firm and all equally willing to have a family; however, differently to the PT approach and in contrast to previous contributions that tend to equalise all women to wives/mothers, we assume that a woman has a family (case M) with probability $\rho \in (0, 1)$ and with probability $(1 - \rho)$ she does not have one (case NM). In other words, the willingness to have a family does not necessarily imply that all women are married and with children. All women are employed and provide a fixed level of time, denoted by l, contractually necessary to perform standard tasks required in their occupation. In addition to time spent in standard tasks, women have an additional unit of time that they can voluntarily spend at work or, if they have one, with their family. If they decide to take on additional projects and tasks, women can receive a financial reward that is increasing in the time (assumed observable) spent at work. Of course, if a woman decides to take on additional work projects, she will have less time to spend with the family, if she has one. We further expand the PT classification of *adaptive* women assuming that there are two types of women in our population: *family-oriented* (FO hereafter) and *career-oriented* (CO hereafter). *Family-oriented* women are employed and experience a positive utility from market activities, however they are naturally more inclined to spend additional time with the family. *Career-oriented* women may have a family and experience a positive utility from spending time at home, but are more naturally inclined to spend time performing additional work tasks.[5]

[5]Notice that we are assuming that the probability of having a family ρ is independent of the particular type of woman considered. This is clearly a simplification. However, it is not obvious whether one type or the other should be assumed to be more likely to have a family. Indeed, the fact that a woman may be family-oriented does not necessarily imply that she will

Let us assume that a fraction $r \in [0,1]$ of the population is composed by family-oriented women (f), while the fraction $(1-r)$ represents the group of career-oriented women (c). We denote by $l^f \in [0,1]$ and $(1-l^f)$ respectively the extra time spent at work and with the family of a family-oriented woman. Similarly, a career-oriented woman will allocate time choosing $l^c \in [0,1]$ and $(1-l^c)$.
Let us suppose that the firm that employes these women offers a remuneration, R^i, $i = f, c$, made by a fixed part a and a bonus Pl^i[6] that depends on the time employees spend performing additional tasks. Then,

$$R^i = al + Pl^i. \tag{4.1}$$

For each type (f or c) depending whether she has a family or not, a woman will incur different costs related to the tasks that she decides to perform during her additional time. In particular, as in Lamantia and Pezzino (2016), we assume quadratic cost functions and with probability ρ (case M) costs are as follows:[7]

$$\begin{aligned} C_M^f(l^f) &= \frac{\alpha\,(l_M^f)^2 + (1-l_M^f)^2 + l_M^f(1-l_M^f)}{2}; \\ C_M^c(l^c) &= \frac{(l_M^c)^2 + \delta\,(1-l_M^c)^2 + l_M^c(1-l_M^c)}{2}, \end{aligned} \tag{4.2}$$

otherwise, in the case NM, women face the following costs:

$$C_{NM}^f(l^f) = \frac{\alpha\,(l_{NM}^f)^2}{2}; \quad C_{NM}^c(l^c) = \frac{(l_{NM}^c)^2}{2} \tag{4.3}$$

where $\alpha, \delta > 1$ are sensitivity parameters representing the different magnitude of the costs for each type of woman. Because of parameters α and δ a family-oriented woman has a comparative advantage in home related activities, while the career-oriented woman has a comparative advantage in work related tasks.

In addition, we assume that all women have some professional vocation (i.e. they experience a positive utility from spending time at work doing non-standard task)[8] and altruism to family members (i.e. they experience a positive utility from spending time at home performing home related tasks). The way women experience the utility of spending time at work or with the family, however, may be distorted by social interactions and, in particular, the way society rewards or punishes the choice of women to allocate time between work and family. We assume that social norms will depend on r, the particular composition of the two types of women in the population. In other words, a society with a

have a higher probability to have a partner and maybe children compared to a career-oriented woman.

[6] P can be interpreted as the financial incentive in a pay for performance scheme. This is assumed to be public knowledge.

[7] Note that we are implicitly assuming that the cost of providing standard effort at work, l, is the same for all women and, for simplicity, normalised to zero.

[8] See Goldin (2006).

large proportion of family-oriented women (high r) will reward the choice of spending time with family and will judge negatively the decision of married woman to spend additional time at work. We assume that the two types of women are likewise affected by society and both family-oriented and career-oriented women tend to conform to society. Then, the larger is r (i.e. the more women in the population are family-oriented) the larger is the utility they obtain from spending time with the family and the lower is the utility from spending additional time at work.
Thus, we define the following utility functions:

$$\rho : \begin{cases} U_M^f = al + l_M^f P + (1+r)(1-l_M^f) + (2-r)l_M^f - C_M^f \\ U_M^c = al + l_M^c P + (1+r)(1-l_M^c) + (2-r)l_M^c - C_M^c \end{cases} \tag{4.4}$$

$$1-\rho : \begin{cases} U_{NM}^f = al + l_{NM}^f P - C_{NM}^f \\ U_{NM}^c = al + l_{NM}^c P - C_{NM}^c \end{cases} \tag{4.5}$$

where $(1+r)(1-l_M^f)$ represents the utility (increasing with r) of spending time with family for a family-oriented while $(2-r)l_M^f$ represents the utility (decreasing with r) of spending additional time at work for the same woman. Similarly, $(1+r)(1-l_M^c)$ represents the utility of spending time with family for a career-oriented woman with $(2-r)l_M^c$ represents the utility of spending additional time at work for the same woman.[9] Notice that women who have no family do not experience the effects of a social stigma. This is in line with the analysis of the data, based on a series of international surveys and the reports of the Eurobarometer[10], analysed in Hakim (2000).

The maximisation of the utility functions in (4.4) and (4.5) defines the following optimal choices for women in terms of time to devote working:

$$(l_M^f)^* = \frac{3+2P-4r}{2\alpha}; \ \ (l_M^c)^* = \frac{1+2P-4r+2\delta}{2\delta}; \ \ (l_{NM}^f)^* = \frac{P}{\alpha}; \ \ (l_{NM}^c)^* = P. \tag{4.6}$$

The optimal times in scenario (NM) are positive for every parameterisation, while the optimal times in the case (M) are positive under the conditions $1+2P < 4r < 3+2P$ and $0 < P < \frac{3}{2}$.[11]
Notice that, not surprisingly, career-oriented women allocate more time at work than family-oriented women. Moreover, we have that:

$$\frac{\partial (l_M^f)^*}{\partial \alpha} < 0, \frac{\partial (l_M^c)^*}{\partial \delta} > 0$$

Intuitively, an increase in the pay-for-performance payment P has a positive effect on the time spent at work for all women. An increase in α increases the

[9] The amount $(2-r)$ is assumed for simplicity in describing the role of social norms in the definition of utility functions. Indeed, such form allows to avoid negative values in the positive component of the utility.
[10] See http://ec.europa.eu/commfrontoffice/publicopinion/index.cfm
[11] These conditions will be taken into account in the following Propositions.

cost of spending time at work for the family-oriented woman and consequently reduces l_M^f; an increase in δ increases the comparative advantage in work related activities of the career-oriented woman and therefore has a positive effect on l_M^c.

In addition $\frac{\partial (l_{NM}^f)^*}{\partial \alpha} < 0$ and $\frac{\partial (l_{NM}^c)^*}{\partial \delta} = 0$ for every parameterisation. The intuition of the effect of α on l_{NM} is the same as for the women with family. Not having a family, however, implies that career-oriented women do not experience a comparative disadvantage in spending time at home.

4.2.1 Static analysis

By substituting the optimal time values of each type of woman in (4.4) and (4.5), we can derive the expected utility of family oriented women, at time t, as:

$$\mathbb{E}[U_f] = \rho\, U_M^f \left((l_M^f)^* \right) + (1-\rho) U_{NM}^f \left((l_{NM}^f)^* \right), \tag{4.7}$$

while career oriented women, at time t, has the following expected utility

$$\mathbb{E}[U_c] = \rho\, U_M^c \left((l_M^c)^* \right) + (1-\rho) U_{NM}^c \left((l_{NM}^c)^* \right). \tag{4.8}$$

The assumption of identical agents allow us to meaningfully define the so called score function, i.e. the difference between the two expected utilities, as follows.

$$y(r) = \mathbb{E}[U^f] - \mathbb{E}[U^c]. \tag{4.9}$$

Expected utilities from the two behaviours become equal for a fraction r^*, when $y(r) = 0$, from which we derive the set of possible equilibria for the system

$$r^* = \left\{ 0, 1, \, \frac{1}{4} \, \frac{\pm 2\sqrt{A} + \left((2\,\delta + 1 + 2\,P)\,\alpha - (2\,P + 3)\,\delta \right) \rho}{\rho\, (-\delta + \alpha)} \right\} \tag{4.10}$$

where $A = \rho\, (\alpha - 1)\, \delta\, \left(\left(\left(\alpha - P^2 \right) \rho + P^2 \right) \delta + \left(\left(P^2 - 1 \right) \rho - P^2 \right) \alpha \right)$.

The non-trivial inner equilibria $r_{1,2}^* = \frac{1}{4} \frac{\pm 2\sqrt{A} + ((2\,\delta+1+2\,P)\alpha - (2\,P+3)\delta)\rho}{\rho\,(-\delta+\alpha)}$ are defined if and only if $A > 0$ and they have to belong to the interval $(0, 1)$. Then, by introducing the following threshold values

$$\begin{cases} \alpha_1 = \frac{(3+2P)^2 \delta}{(1+2P)^2 + 8(1+P)\delta}; \\ \alpha_2 = \frac{(1-2P)^2 \delta}{(3-2P)^2 + 8(-1+P)\delta}; \\ \rho_1 = \frac{4P^2(\alpha-1)\delta}{-(1+2P)^2\alpha + (9 - 8\alpha + 4P(3+(P-2)\alpha))\delta}; \\ \rho_2 = \frac{4P^2(\alpha-1)\delta}{-(3-2P)^2\alpha + \delta - 4P\delta + 4(2+(-2+P)P)\alpha\delta}, \end{cases} \tag{4.11}$$

the following Proposition holds:

Proposition 24 *Let consider the inner equilibria called* $r_{1,2}^*$. *Then, the following cases arise:*
(a) If $\alpha < \delta$, *then there exist one or two inner equilibria. In particular, we have that:*

- *(a1) there exists* $r_1^* \in (0,1)$ *if* $1 < \alpha < \alpha_1$ *and* $\rho_1 < \rho < 1$*;*
- *(a2) there exists* $r_2^* \in (0,1)$ *if* $1 < \alpha < \alpha_2$ *and* $\rho_2 < \rho < 1$*;*
- *(a3) there exist both* $r_1^* \in (0,1)$ *and* $r_2^* \in (0,1)$ *if* $1 < \alpha < min(\alpha_1, \alpha_2)$ *and* $max(\rho_1, \rho_2) < \rho < 1$.

(b) If $\alpha > \delta$ *and* $B < \rho < 1$*, then no or a single (*r_2^**) inner equilibrium exists. In particular, we have that* $r_2^* \in (0,1)$ *exists if* $1 < \alpha < \alpha_2$ *and* $\rho_2 < \rho < 1$.

Before investigating the dynamic stability of the system, it may be interesting to discuss some results deriving from simple comparative statics analysis. First, by studying the variation of the value assumed by $r_{1,2}^*$ as α or δ vary, we get some counter-intuitive results. Indeed, we have that (i) $\frac{\partial r_1^*}{\partial \alpha} < 0$ and (ii) $\frac{\partial r_1^*}{\partial \delta} > 0$, while (iii) $\frac{\partial r_2^*}{\partial \alpha} > 0$ and (ii) $\frac{\partial r_2^*}{\partial \delta} < 0$. Then, although an increase in the intrinsic cost of working for a FO woman induces a decrease in her optimal working time, The result (i) implies that when α increases the probability to be a family-oriented woman decreases. Symmetrically, even though an increase in the intrinsic cost of staying with family for a CO woman induces an increase in her optimal working time, The result (ii) implies that when δ increases the probability to be a family-oriented woman increases. In addition, we have that $\frac{\partial r_1^*}{\partial \rho} > 0$ ($\frac{\partial r_2^*}{\partial \rho} < 0$) and this means that, intuitively, the probability to have a family (that is the case M) positively affects the probability to be a family oriented woman. Finally, we get that the interior equilibrium has a non-monotonic relationship with the payment released by the firm for the bonus (P). Indeed, we have that $\frac{\partial r_1^*}{\partial P} < 0$ if and only if $P > \frac{\sqrt{A}}{\delta(\alpha-1)(1-\rho)}$.

4.3 Evolutionary analysis (word of mouth)

In this section we endogenise r assuming that, at time t, the probability to be a FO woman is given by the fraction r of family-oriented women in the population at that time. Therefore, the probability to be family-oriented or career-oriented is updated according to the expected utilities from the two possible positions. We model the dynamics of the probability of being family oriented by the word of mouth evolutionary framework (see Dawid, 1999). Indeed, we suppose that at each t two women working in the company (at the same salary conditions) meet and compare their positions (*or inclinations*). Clearly, if both women have the same inclination (either both are FO or CO) they have the same utility (payoff) and no switching mechanism arises. Differently, if a family-oriented woman meets a career-oriented one (the opposite is equivalent), they may reconsider their position (and then the behaviour until time $t+1$) according to the utility achieved by the other. Thus, a FO woman may change her mind if she meets a CO woman or viceversa. Obviously, the higher the difference is

between FO and CO's payoff, the more likely one of the two womem will be inclined to switch to the other position.
In order to describe the switching mechanism behind the model, we define as Φ the probability to switch from being CO to FO, given that $U_f \geq U_c$. Then, we have:

$$\Phi(U_f - U_c) = \mathbb{P}(c \to f | U_f \geq U_c) \tag{4.12}$$

where $\Phi : \mathbb{R} \to [0, 1]$, being a probability distribution function, is non decreasing in y, with $y = U_f - U_c$. Moreover, it holds that

$$\lim_{y \to -\infty} \Phi(y) = 0; \quad \lim_{y \to +\infty} \Phi(y) = 1.$$

Therefore, we can rewrite the probability that a CO woman becomes a FO as

$$p_{c \to f} = r\Phi(y) \tag{4.13}$$

where r represents the probability that a CO woman meets a FO and $\Phi(y)$ the probability to change inclination.
Differently, being $(1 - r)$ the probability that a FO woman meets a CO and $\Psi(-y) = 1 - \Phi(y)$ the probability to change inclination, the overall probability that a FO woman becomes CO is given by

$$p_{f \to c} = (1 - r)(1 - \Phi(y)) \tag{4.14}$$

Assuming that the matching between women in the company (our *population*) is uniform and that a sufficiently large number of sampling is taken into account, the average payoff difference of the two inclinations is approximated by the expression in (4.9).
We consider from now that the fraction of FO is denoted by r_t and assume that the expected payoff (or utilities) are functions of r_t (consequently, $y = y(r_t)$), the dynamics of r_t is defined by the following map:

$$r_{t+1} = r_t + (1 - r_t)p_{c \to f} - r_t p_{f \to c} \tag{4.15}$$

where $(1 - r_t)p_{c \to f}$ represents the share of career oriented women becoming family oriented while $r_t p_{f \to c}$ denotes the share of women making the reverse path.
By substituting expressions (4.13) and (4.14) in the equation (4.15) and algebraic manipulation, we can rewrite the map as:

$$F := r_{t+1} = r_t[1 + (1 - r_t)G(y(r_t))] \tag{4.16}$$

where $G(y(r_t)) = \frac{2}{\pi} \arctan(\frac{\lambda}{2}\pi \, y(r_t))$ (see Bischi et al., 2009b). In this specification, λ represents the positive parameter governing the intensity of choice, that is the willingness to change behaviour from being family oriented to career oriented.

According to Lamantia and Pezzino (2017), the function G that derives from a distribution function as described above, gets the following properties:

$$\lim_{y \to -\infty} G(y) = -1;$$
$$\lim_{y \to +\infty} G(y) = 1.$$

In addition, assuming unimodality and symmetry of the density function associated to Φ, the following assumptions on $G(y)$ hold:

- $G(0) = 0$;
- G is symmetric with respect to 0;
- G is increasing;
- G is convex in the range $(-\infty, 0)$ and concave in the range $(0, +\infty)$;
- G is differentiable at least in $y = 0$.

The map (4.16), modelling how the fraction of family-oriented women changes over time, admits the two boundary trivial fixed points $0, 1$ and up to two interior equilibria $r^*_{1,2} \in (0, 1)$, such that $y(r^*) = 0$. As described in Lamantia and Pezzino (2017), we can notice that the function G is monotone and then the sign of G and $y(r_t)$ coincides. This means that an increase in the level of the share r occurs if and only if $y > 0$ and then the map represent an example of monotone selection dynamics (see Cressman and Ansell, 2003; Weibull, 1995).

4.4 Dynamic analysis

In this section, we focus on analysing the dynamic properties of the discrete time map (4.16), which describes the evolution over time of the fraction of family-oriented women. The map admits two types of equilibria: boundary equilibria $r_c = 0$ and $r_f = 1$ and inner equilibria r^* which satisfy the equality $y(r^*) = 0$. The boundary equilibria exist for all parameterisations and describe monomorphic configurations of the population in which all women are career-oriented ($r_c = 0$) or family-oriented ($r_f = 1$). Instead, the inner equilibria correspond to polymorphic configurations of the population in which both family-oriented and career-oriented exist. By recalling the statement in Proposition 24, we have that zero, one or two inner equilibria (r^*_1 and r^*_2, respectively) may exist. Then, the analysis moves on the investigation about the convergence to the trivial equilibria when no inner equilibria exist and the stability of the inner equilibria when they exist. In this regard, the following proposition holds:

Proposition 25 *Let consider the map defined in (4.16) and the conditions stated in Proposition 24 on the existence of the inner equilibria.*
The inner fixed point r^*_1 *is asymptotically stable if* $\lambda \in \left(0, \frac{\alpha\delta}{r^*_1(1-r^*_1)\sqrt{A}}\right)$. *Contrariwise, the inner fixed point* r^*_2 *is always unstable.*

Proof. Evaluating the first derivative of F in the interior fixed point r^*_{int}, we have

$$F^{'}(r_1^*) = (1 + (1 - r_1^*)G(0)) + r_1^*((1 - r_1^*)G^{'}(0) - G(0)) \tag{4.17}$$

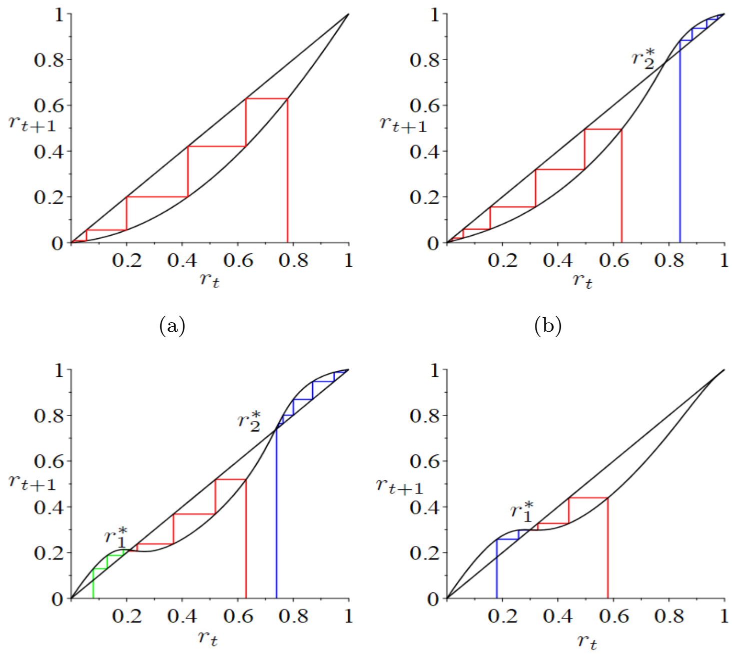
From the property $G(0) = 0$, we have that the first derivative in the fixed point reduces to be

$$F^{'}(r_1^*) = 1 + r_1^*((1 - r_1^*)G^{'}(0) \tag{4.18}$$

By calculating the value of $G^{'}(0)$, we obtain that $G^{'}(0) = -\frac{2\lambda\sqrt{A}}{\alpha\delta}$. Then, by substituting such value in (4.18) and recalling that the fixed point loses its stability when $|F^{'}(r_1^*)| = 1$, we derive that $F^{'}(r_1^*) = 1$ for $\lambda = 0$ and $F^{'}(r_1^*) = -1$ for $\lambda = \frac{\alpha\delta}{r_1^*(1-r_1^*)\sqrt{A}}$. Then, the result follows. Analogously, evaluating the first derivative of the map in r_2^*, it is straightforward to show that an interval in which such fixed point is locally stable does not exist. ■

Remark 14 *At* $\lambda = \frac{\alpha\delta}{r_1^*(1-r_1^*)\sqrt{A}}$*, the inner equilibrium* r_1^* *looses its stability through a period doubling bifurcation and then, as the parameter* λ *increases, cyclic or chaotic regimes appear.*

The following Panels in Figure 4.1 furnish a numerical confirmation of the scenarios described by Proposition 24 and 25. Panel (a) in Figure 4.1 shows a scenario in which only the boundary equilibria are admitted and, for all initial conditions r_0, almost all trajectories (an example is given by the red line) converge to the equilibrium 0. This means that, in this case, every initial condition on the fraction of FO women leads to a final state in which the population is monomorphic and composed only by CO women. Panel (b) in Figure 4.1 describes the case in which only the trivial equilibria and the unstable inner equilibrium r_2^* are admitted. As the graph suggests, an initial condition on the left of r_2^* leads to the final state with all CO women (red line) while an initial condition on the right of r_2^* leads to the final state with all FO women (blue line). Hence, the unstable fixed point r_2^* separates the basins of attraction of the boundary equilibria. Panel (c) in Figure 4.1 show a numerical example in which, in addition to the boundary equilibria, both r_1^* and r_2^* exist. In the graph we can notice that both initial conditions on the left and on the right of r_1^* (see red and green lines) lead to the final state r_1^* while initial conditions on the right of the unstable equilibrium r_2^* leads to the boundary equilibrium 1 where only FO women are present. Finally, Panel (d) in Figure 4.1 shows the case in which every initial condition leads to the stable equilibrium r_1^*.



(a) (b) (c) (d)

Figure 4.1: Different behaviours of the map for different parameter sets. (a) Parameter set: $P = 1.45, \alpha = 1.45, \delta = 2.16, \rho = 0.75, \lambda = 12$. The map admits only the boundary equilibria 0 (stable) and 1 (unstable). (b) Parameter set: $P = 0.65, \alpha = 1.25, \delta = 2.16, \rho = 0.75, \lambda = 12$. The map admits a unique inner equilibrium r_2^* (unstable) and the stable boundary equilibria. In this case r_2^* divides the basins of attraction of 0 and 1. (c) Parameter set: $P = 0.65, \alpha = 1.1, \delta = 2.16, \rho = 0.75, \lambda = 20$. The map admits the boundary equilibria (0 is unstable while 1 is stable) and two inner equilibria r_1^* (stable) and r_2^* (unstable). (d) Parameter set: $P = 1.05, \alpha = 1.128, \delta = 2.1, \rho = 0.75, \lambda = 45$. The map admits the boundary equilibria (they are both unstable) and the inner equilibria r_1^* (stable).

4.4.1 The crucial role of the intensity of choice

From a dynamic point of view, a crucial role is played by the intensity of choice parameter λ. Indeed, we can notice how, as λ varies, the nature of the fixed point r_1^* changes and more complex regimes may arise.
Cobweb plots in Panels of Figure 4.2 describe different scenarios as the intensity of choice parameter varies. By assuming the following Parameter set $P = 1.45, \alpha = 1.18, \delta = 3.16, \rho = 0.75$ and leaving to vary the parameter λ, in Panel (a) we notice that, starting from an initial condition r_0 and $\lambda = \lambda_{st} = 10$, after some iterations the system converges to the stable r^{*_1}. By increasing λ to $\lambda_{cy} = 60$, Panel (b) describes the case in which a period doubling bifurcation is occurred and a stable 2-cycle appears. Finally, a really larger value for λ, $\lambda = \lambda_{ch} = 210$, Panel (c) depicts the arise of a chaotic regime for the system.

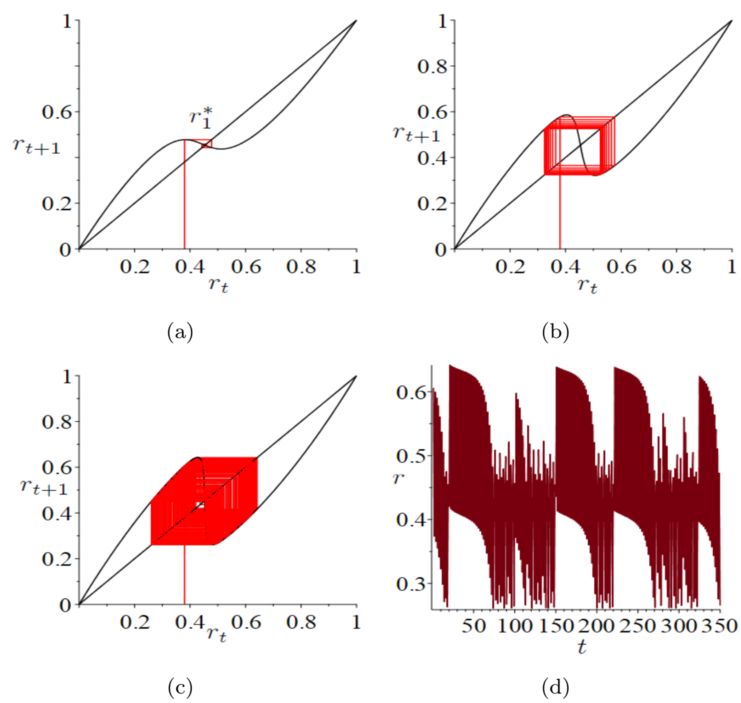


Figure 4.2: Different scenarios as λ varies, starting from $r_0 = 0.38$. (a) Convergence to the stable r_1^* when $\lambda = \lambda_{st} = 10$. (b) Convergence to a stable 2-cycle when $\lambda = \lambda_{cy} = 60$. (c) Presence of a chaotic regime for $\lambda = \lambda_{ch} = 210$.

The destabilising role of λ can be described also by the bifurcation diagram in Figure 4.3. Indeed, the graph confirms the observation in the Remark 14 and we can notice that as λ increases until its value violates the stability condition (see Proposition 25), a period doubling bifurcation occurs and a 2-cycle appears. Then, as the intensity of choice further increases, a period doubling cascade occur and finally a chaotic regime arises. The scenarios described by the Figures confirm what is mentioned in the literature. Indeed, several works on different topics, show how the assumption of a high value of the intensity of choice, that is a high willingness of the agents to change their *beliefs* (in our model we define them as *inclinations*) (see Brock and Hommes, 1997). With regard to our model, this means that as the willingness of women in changing their inclination between being *family-oriented* and *career-oriented* increases, the composition of the population will first tend to converge towards an inner equilibrium in which FO and CO women coexist. Then, as the intensity of choice assumes higher values, the fraction of FO women in the population cyclically reach values around the inner equilibrium and at the end, when λ is sufficiently high, it becomes unpredictable and complex regimes arise in the system.[12]

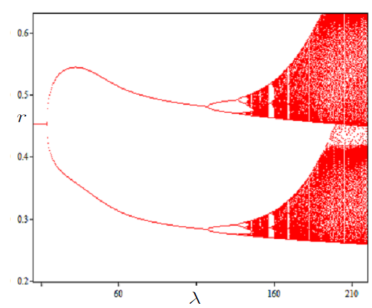
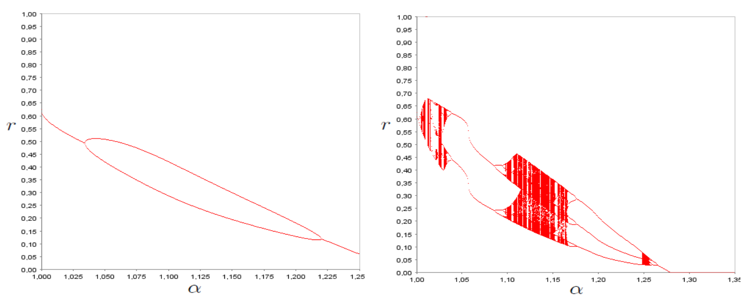


Figure 4.3: Parameter set: $P = 1.45, \alpha = 1.18, \delta = 3.16, \rho = 0.75$. Bifurcation diagram with respect to λ, starting from the initial condition $r_0 = 0.38$.

[12]The analysis performed in this section has regarded the case in which a unique interior fixed point exists. This choice allowed us to show some socially relevant dynamic phenomena related to how the parameter affects the nature of the unique interior equilibrium. The analysis of the effects of λ in cases in which both the inner equilibria exist represents one the strands of the investigation that may be furtherly deepened.

4.4.2 The effect of intrinsic costs

A fundamental role in defining the dynamics of the population considered in our model is then played by the parameters of intrinsic costs that the different types of women experience, α and δ, respectively. In the previous sections, we have seen how social norms homogeneously affect the utility functions of the two types of women. In this section, we can notice that the variation of some of these intrinsic costs can be able to dominate the effect of social norms. Indeed, although the social norms act in favour of family-oriented women, an increase in the value of α, and therefore in the intrinsic cost attributed by being family-oriented to the extra working time, along time ends up in reducing the fraction of FO women. This means that they will then prefer to radically change their inclinations because they consider excessively inconvenient, in the society that surrounds them, to have an inclination towards creating or taking care of the family.
Figure 4.4 shows a numerical example of this phenomenon and the connection between the role of α and the increase in the willingness to change inclinations (defined by λ). When λ is sufficiently low (that is, $\lambda_l = 20$), the graph in Panel (a) allows us to notice that, as α increases, (i) the value of the stable inner equilibrium decreases until $\alpha_{pd} \simeq 1.033$ where it undergoes a period doubling bifurcation; (ii) a stable 2-cycle arises in the system until $\alpha_{ph} \simeq 1.223$ where it undergoes a period halving bifurcation allowing to recover the stability of the inner equilibrium. When λ is fixed as sufficiently high ($\lambda_h = 70$), an increase of α may allow observing periodic cycles and the emergence of chaotic behaviours, as depicted in Panel (b).

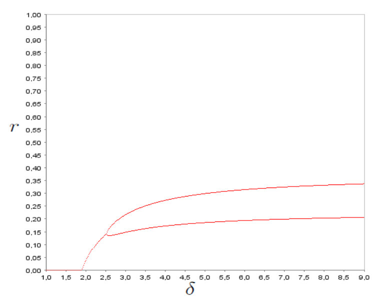


(a) (b)

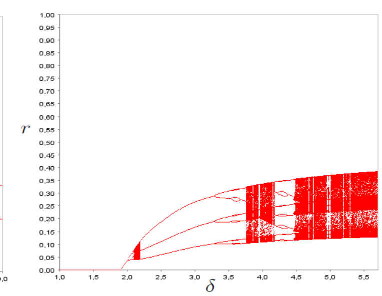
Figure 4.4: (a) Parameter set: $P = 1.45, \delta = 3.16, \rho = 0.75, \lambda_l = 20$. The Bifurcation diagram with respect to α depicts (i) the loss of stability for the inner equilibrium through a period doubling bifurcation and then (ii) the recover of the stability through a period halving bifurcation, as α increases. (b) Parameter set: $P = 1.45, \delta = 3.16, \rho = 0.75, \lambda_h = 70$. The bifurcation diagram depicts the occurrence of period cycles and then windows of chaotic behavior, as α increases.

In the opposite way, (i) the positive effect of social norms on the time spent at home and (ii) an increase in the value of δ, that is in the intrinsic cost attributed by being career-oriented to the time spent at home, along time ends up in increasing the fraction of FO women. This means that some CO women will then prefer to radically change their inclinations because they consider excessively inconvenient, in the society that surrounds them, to have an inclination towards preferring commitment to work.

Figure 4.5 numerically shows this phenomenon and the arise of complexity when an increase in δ is associated with high levels in the willingness to change inclinations (defined by λ). Indeed, for a low value of λ (that is, $\lambda_l = 20$), we can notice that, as δ increases, (i) the value of the stable inner equilibrium increases until $\delta_{pd} \simeq 8.04$ where it undergoes a period doubling bifurcation and, for $\delta > \delta_{pd}$, a 2-cycle arises in the system. When λ is fixed as sufficiently high ($\lambda_h = 70$), an increase of δ may allow to observe the occurrence of periodic cycles and then the arise of chaotic behaviours, as depicted in Panel (b).



(a)



(b)

Figure 4.5: (a) Parameter set: $P = 1.45, \alpha = 1.18, \rho = 0.75, \lambda_l = 20$. The Bifurcation diagram with respect to δ depicts the loss of stability for the inner equilibrium through a period doubling bifurcation and the occurrence of a 2-cycle, when λ is sufficiently low. (b) Parameter set: $P = 1.45, \alpha = 1.18, \rho = 0.75, \lambda_h = 70$. Occurrence of periodic cycles and arise of chaotic behaviours, for a high value of λ.

4.5 Conclusions

In this chapter, by adopting the point of view described by the PT, we have employed an evolutionary framework to study the short and long run decisions of women concerning the allocation of time between work and family. Specifically, we have assumed a population with two types of women: family-oriented (FO), and career-oriented (CO). The preferences of both types of women are affected by extrinsic benefits (e.g. a Pay-for-Performance contract at work), intrinsic costs (i.e. their inclination toward spending time at work or with the family) and by social norms. We have assumed in addition that women socially interact (according to *word of mouth* dynamics) and compare their different positions, learning about possible payoff differentials. , therefore, sparks the evolution of the distribution of types of women (and corresponding behaviour) in the population.

Our findings show that the social interaction of women in the workplace and the intrinsic costs they have being *family-oriented* or *career-oriented* may lead to both scenarios in which the population converges to monomorphic configurations and scenarios where the population converges to a polymorphic composition (that is, a stable inner equilibrium for the system may exist) are achievable. Moreover, due to the highly destabilising role of the intensity of choice parameter and the interplays between social norms and intrinsic costs, both periodic cycles and chaotic regimes may emerge. The latter therefore make the fluctuations in the composition of the female population erratic and unpredictable over time.

Clearly, the model proposed in this chapter represents a strong stylisation of the possible approaches with which the role of women in society, in work as well as in the family, can be modelled and analysed. By maintaining (i) the idea that the female population is composed of women with different but *adaptive* inclinations and (ii) that social norms and interactions with other subgroups of the population may generate switching in the inclinations, possible future extensions and research may go in different directions. One of the future researches will be the analysis of how the effects of social norms and interaction on the women's short and long term decisions may reflect possible policies aimed at the one hand at encouraging female labour participation and on the other hand at fostering women in spending time with family. Another research question will concern the matching of the female population (with its different inclinations) with the male population (with its different inclinations) and then (i) the analysis of how the interaction, in the workplace, of agents with different sex and then with different inclinations can affect the long-term composition of the two populations and (ii) the investigation of how it can be managed the problem of gender salary gap between male and female workers.

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