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NONLINEAR DYNAMICS AND ECONOMIC GROWTH. THE INFLUENCE OF ELASTICITY OF SUBSTITUTION BETWEEN INPUT FACTORS AND DIFFERENTIAL SAVINGS PROPENSITIES.

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Abstract

This thesis investigates the qualitative and quantitative dynamics of the Solow-Swan growth model with differential saving considering different production functions in order to analyse how the long run behaviour of the economy is influenced by the elasticity of substitution between production factors and by different savings propensity between workers and shareholders. In the first chapter the economic growth problem of establishing a relation between the elasticity of substitution, capital and output per-capita levels when dealing with a non constant elasticity of substitution production function is discussed. Starting from a discrete-time setup, some definitions of elasticity of substitution associated to an attractor are proposed and a method to measure it is suggested. The main goal is to compare dynamic growth models with VES, sigmoidal and CES production functions. To this end, the method proposed is applied to the Kaldors model using a VES production function with constant returns to scale. It is found that when simple dynamics are exhibited, a country characterized by production functions with higher elasticity of substitution experiences higher capital and output per-capita equilibrium levels. On the other hand, when the long term dynamics consist of cycles or more complex features, then an ambiguous relation between elasticity of substitution and asymptotic dynamics is shown. In the second chapter the Kaldor growth model is analysed, assuming the Shifted Cobb-Douglas (SCD) production function, a technology that - differently from CES and VES one - allows one to consider the dynamics of non developed and developing countries as well as that of developed economies. The resulting model is a discontinuous map generating a poverty trap. Furthermore multistability phenomena may emerge: next to the vicious circle of poverty, long run behaviours may include boom and bust periods (fluctuations may arise when the elasticity of substitution is lower than one) and convergence to a positive level of capital per-capita. In the last chapter the discrete time neoclassical one-sector growth model with differential savings is studied assuming the Kadiyala production function which shows a variable elasticity of substitution symmetric with respect to capital and labor. It is shown that, if workers save more than shareholders, then the growth path is bounded from above and the boundary is independent from of the savings rate of shareholders. The growth path for non-developed countries is influenced only by the savings rate of shareholders while level of capital per capita of developed economies is influenced by the savings rate of workers. Moreover, multistability phenomena may occur so that the model is able to explain co-existence of under-developed, developing and developed economies. Fluctuations and complex dynamics may arise when the elasticity of substitution between production factors is lower than one and shareholders save more than workers.

1 Introduction

In the classic article *A Contribution to the Theory of Economic Growth* [67], the Nobel Prize-winning Robert M. Solow investigated the relationship between the structure of production functions and income distribution. Solow proposed a model describing the dynamics of the physical capital and the long-term evolution of the growth process taking into consideration the role of capital, labour and technology. In his essay he took into consideration how the long-run equilibrium or disequilibrium of the economy changes considering different types of production functions: the Harrod-Domar, the Cobb-Douglas (CD) and a third type of production function that five years later had been generalized with the two-factor Constant Elasticity of Substitution (CES) production function (see Solow *et al.* [4]). He refuted the Harrod-Domar assumption of fixed proportions (see Harrod [32] and Domar [28]) and supposed the possibility of substituting labour for capital in production (see Solow [67]). This assumption has led the way to investigations of how the elasticity of substitution affects capital and output equilibrium levels and hence economic growth. When the Cobb-Douglas production function is considered, the model monotonically converges to the steady state, since the elasticity of substitution between production factors is constant and equal to one.

Notice that elasticity of substitution between production factors σ measures how quickly the marginal rate of technical substitution of labour for capital changes as we move along an isoquant. The greater the ease with which one factor can be substituted for another (for a given level of output), the greater will be the elasticity of substitution. In linear production functions inputs are perfectly substitutable for each other, isoquants are straight lines and $\sigma = +\infty$. On the contrary, in fixed-proportions production functions inputs are perfect complements, isoquants are L-shaped and $\sigma = 0$. Many papers investigating neoclassical growth model used the CD specification of the production function in which capital and labour can be substituted for each other and the elasticity of substitution is equal to one. More recently, several contributions investigated theoretically and empirically the role played by the CES production functions (see Klump and Preissler [43], Klump and de La Grandville [42], Miyagiwa and Papageorgiou [53] and Masanjala and Papageorgiou [47]) in which elasticity of substitution between inputs is constant and takes values that are either greater or lower than one. Although CES production functions widen the range of values of the elasticity of substitution from 0 to ∞, these production functions restrict σ to be constant along an isoquant whereas the elasticity of substitution between inputs should be a variable depending upon output and factor combinations (see Hicks [33], Allen [2] and Revankar [63]). Moreover, for more than two factors, different degrees of substitutability between inputs are not allowed (see the Impossibility theorem of Uzawa [75] - McFadden [48]). The class of Variable Elasticity of Substitution (VES) production functions proposed by Lu and Fletcher [45], Revankar [62] and Sato and Hoffman [66] fix this criticisms exhibiting an elasticity of substitution between capital and labour that is affected by changes in the economy's per-capita capital level. Many studies analyzed the role of a variable elasticity of substitution within the Solow model (see Karagiannis *et al.* [40], Papageorgiou and Saam [55]).

In 1989 de La Grandville considered the Solow model with a normalized CES function (equal to a in the third case presented in Solow's work) and showed that an higher elasticity of substitution implies an higher capital per-capita level and he conjectured that the huge growth in Japan and East Asian countries could had been due to an higher elasticity of substitution between capital and

labour instead of a more efficient technical progress or an higher savings rate. Rainer Klump and Olivier de La Grandville considered a Solow type growth model and a normalized CES production function and demonstrated that an economy with higher elasticity of substitution experiences a higher level of per-capita income, both in transition and in steady state (Klump and La Grandville [42]). They compared economies characterized by the same growth model and CES production function, differentiated only by the degree of elasticity of substitution.The same result was found by Klump and Preissler [43].
In line with these researches Miyagiwa and Papageorgiou used the CES production function in the Diamond overlapping-generation model (see Diamond [26]) to study economic growth and its relation with elasticity of substitution between production factors (see Miyagiwa and Papageorgiou [53]). Differently from the other works, they found that, if capital and labour are relatively substitutable, an higher elasticity of substitution between production factors leads to a lower output per worker, both in transition and in steady state. They concluded that whether the economic growth is positively or negatively affected by the elasticity of substitution between production factors it depends on the used setup, i.e. the Solow or the Diamond framework.

Recently several papers have considered the Solow-Swan model with Constant Elasticity of Substitution (CES) or the Variable Elasticity of Substitution (VES) production functions, in order to analyze the long-run dynamics of the system when the elasticity of substitution is lower then one, greater then one or even non constant (for CES see Brianzoni et al. [14, 18], Masanjala and Papageorgiou [47] and Papageorgiou and Saam [55] while for VES see Brianzoni et al. [19] and Karagiannis et al. [40]). Most of the cited works found that fluctuations and even more complex dynamics may arise if the elasticity of substitution is sufficiently low. Evidently the elasticity of substitution between production factors plays a crucial role in the theory of economic growth. Moreover it represents one of the determinants of the long-run equilibrium level (for the correlation between elasticity of substitution and capital per-capita levels see Klump and La Grandville [42] and Miyagiwa and Papageorgiou [53]).

It is easy to see that accurate connection between endogenous economic growth and elasticity of substitution can be determined when a framework with a CES production function is proposed. In this case it is possible to investigate both long run dynamics and the relationship between elasticity of substitution and economic growth. Differently, with Variable Elasticity of Substitution production functions such as the VES production function or the sigmoidal one, even if the qualitative and quantitative long term dynamics have been widely studied, no attention has been paid to the relation between variable elasticity of substitution and growth. This limitation is due to the fact that with VES or other non-constant elasticity of substitution production functions the elasticity itself depends on the level of capital per-capita. In fact, once we define the elasticity of substitution as a variable depending on capital or output, the methodology used with CES production function to investigate its relationship with economic growth loses effectiveness. More precisely, with CES production functions one investigates how the elasticity of substitution influences the long term capital per-capita levels. Differently, with VES or sigmoidal production functions this relation is altered, since the same elasticity of substitution is affected by the capital per-capita levels. Hence, we pass from a unilateral affect system ($\sigma \to k_t$) to a bilateral affect system ($\sigma \leftrightarrow k_t$), where σ is the elasticity of substitution between production factors while k_t represents the capital per-capita level at a given time $t \in \mathbb{N}$.

The choice to investigate the behaviour of the growth model considering a Variable Elasticity of Substitution production function is due to the fact that many empirical studies prove that VES (instead of CES) production functions are a better representation of reality. In 1968, Lovell [44] rejected both the Cobb-Douglas and the CES specifications in favor of the VES production function using data for two-digit U.S. manufacturing industries; the same result was obtained by Diwan [27] for individual U.S. manufacturing firms, by Revankar [63] for the private non-farm sector of the U.S and by Meyer and Kadiyala [50] using agricultural data. Evidences in favor of the VES production function are also provided for Japanese (see Bairam [8] and Sato and Hoffman [66]) and Soviet (see Bairam [7]) economies, as well as for larger region data (see Karagiannis et al. [40]). All these contributions uphold that the VES production function is a better representation of the elasticity of substitution.

As a further step on the economic growth theory Kaldor ([38, 37]) proposed a Solow's type growth model in which the two income groups (labor and capital) might have different savings behaviour. Consequently the investigation of the influence of differential savings rates between workers and shareholders arises in literature. Böhm and Kaas [12] studied the Kaldor model assuming a generic production function satisfying the weak Inada conditions and showed that instability, fluctuations and complex dynamics may emerge. Recently Brianzoni et al. [14, 15, 18] investigated Kaldor's growth model in discrete time with differential savings and endogenous labour force growth rate while assuming a CES production function. They found that the model can exhibit cycles or even chaotic dynamic patterns. Moreover, Cheban et al. [22] investigated the neoclassical growth model with the labour force dynamics described by the Beverton-Holt equation (see [9]) assuming a CES production function. In both contributions the authors found that if the elasticity of substitution is positive but sufficiently low, the economic patterns are bounded, and, if the elasticity of substitution is close to zero, the economic system can converge to a steady state characterized by no capital accumulation. Tramontana *et al.* [72] used the Leontief production function and proved that cycles and fluctuation can be exhibited if shareholders save more than workers. As a further step in this field, the role of different VES production functions has been considered: Brianzoni *et al.* [19] studied Kaldor model with Revankar [63] production function; they found that unbounded endogenous growth is possible (differently from CES) and fluctuations may arise if shareholders save more than workers and the elasticity of substitution between production factors falls below one. Similar results can be found considering non-concave production functions (see Brianzoni *et al.*[16] and Michetti [51]).

As Azariadis and Stachurski [6] showed, concave neoclassical growth models don't take in consideration the differences production technology between rich and poor countries while non-concave growth models may generate persistent-poverty aggregate income data. In order to take into account the existence of poverty trap (the condition for which a country need a critical level of physical capital before a growth dynamic could be observed), recently Brianzoni et al. [16] considered a non-concave production function. Also for non developed or developing countries complicated dynamics emerge if the elasticity of substitution is sufficiently low confirming that the elasticity of substitution is responsible for the creation and propagation of complexity.

In the first part of this work a way to establish a relation between the elasticity of substitution between production factors and the long term growth dynamics when dealing with a non-constant elasticity of substitution production function is suggested. While with CES production functions, whatever the attractor, the elasticity between production factors is a constant, with VES or sigmoidal production functions σ depends on the output k_t and hence on the long term dynamics exhibited by the model, fixing a contrast in measuring the output variation depending on the elasticity of substitution. The method introduced will be used to analyze the relation between the elasticity of substitution and capital per-capita levels considering Kaldor's growth model [37] and the Variable Elasticity of Substitution (VES) production function in intensive form with constant return to scale, as given by Revankar (see Revankar [62] and Karagiannis et al. [40]). The purpose is also to verify whether the main result obtained by Klump and La Grandville [42] using the CES function still holds, i.e. if greater elasticity of substitution implies higher equilibrium levels also with VES, so as to extend the study in Brianzoni et al. [19]. It is demonstrated that, when the long run dynamics are simple, then there exists a positive correlation between elasticity of substitution, capital and output per-capita associated to the attractor. On the other hand, when the economic patterns exhibit cycles or more complex dynamics, an ambiguous relation between elasticity of substitution and the asymptotic dynamics is shown.

In the second part of this work the discrete time one-sector Solow-Swan growth model with differential savings as given by Böhm and Kaas [12] is studied while assuming that the technology is described by the Shifted Cobb-Douglas (SCD) production function as proposed by Capasso et al. [20]. As in Brianzoni et al. [16, 17] the use of a non-concave production function states the existence of a poverty trap. Notice that whereas CES and VES production functions well describe developed economies but they are not able to explain dynamics related to non developed countries, the SCD production function implies a minimum level of physical capital essential for production, a requirement of capital needed in order to observe increasing returns. This kind of production function is often considered in literature in order to describe the growth dynamics of developing countries. Indeed, as Azariadis and Stachurski [6] thoroughly explain, poor economies are often characterized by market failure, inefficient practices, "institution failure" and also social norms and conventions which cause the well know "vicious circle of poverty". This considerations make the model economically significant in order to analyze the growth dynamics of developing countries: can a poor economy escape from poverty trap? Which is the required initial investment? If a developing country has passed the poverty trap just now, could it's economy fall down again into it? We shall try to answer these questions.
From the mathematical point of view, when the SCD production function is considered the resulting model is described by a discontinuous map, a type of framework recently considered in several economic models (see, among all, Böhm and Kaas [12] and Tramontana et al. [72, 74, 73]) since recent mathematic tools allow to investigate economic phenomena defined by discontinuous systems. The main goals are to describe the qualitative and quantitative long run dynamics of the growth model and to evaluate the relation between elasticity of substitution and capital per-capita equilibrium levels in not well developed countries. The results of the analysis show that complex dynamics, multistability phenomena and non-connected basin of attraction may emerge. Moreover, as in Klump and La Grandville [42], a positive correlation between elasticity of substitution and long term dynamics is exhibited.

The third part of this work extends previous literature on economic growth by examining the neoclassical one-sector growth model with differential savings while assuming that technology is described by the Kadiyala [36] production function: a VES production function whom property

is to present elasticity of substitution symmetric with respect to input factors, fixing monotony's critic moved to main VES functions. The aim of the work is to investigate how the elasticity of substitution between capital and labour and savings rate of capitalists (shareholders) and workers influence the speed with which economies grow, the existence of poverty traps and the occurrence of fluctuating long run behaviours.
It is found that when the elasticity of substitution between labour and capital is lower than one the growth path for non-developed countries is influenced only from investments made by capitalist while for developed economies the level of capital increase only for higher values of the savings rate of workers.
As in Chakraborty [21] poverty traps may result if savings and investment rates are low despite the absence of inefficient technology, mainly considered the source of "vicious circle of poverty" (see among all Capasso *et al.* [21] and Azariadis and Stachurski [6]).
In addition qualitative and quantitative dynamics of the model are analyzed: multistability phenomena, fluctuation and complex dynamics can be observed if elasticity of substitution is lower than one, confirming results obtained with different technologies.

This work is organized as follow: in the first chapter of this study the definitions of single-value measures associated to an attractor are proposed and a method to measure the elasticity of substitution associated to an attractor is suggested. It is highlighted how this technique make it possible to compare models with VES, sigmoidal and CES production functions. The relation between elasticity of substitution and capital per-capita equilibrium levels considering Kaldor's model with VES production function [62] is investigated by using both analytical tools and numerical techniques; to this aim the new measuring method proposed is applied. In the second chapter the Kaldor model with Shifted Cobb-Douglas is presented and its proprieties are discussed. The existence and the stability of the steady states are analyzed and the possibility of multiple equilibria, complex dynamics and also complex basins are demonstrated. In the third chapter the influence of savings rates on the growth path of the Kaldor model with Kadiyala production function is analyzed. The dynamical behaviour of the framework is investigated and complex dynamics and multistability phenomena are discussed. Chapter 5 conclude the work.

2 Measures of elasticity of substitution and equilibrium levels associated to attractors

2.1 Preliminaries

Consider a discrete time setup in which $k_t = \frac{K_t}{L_t} \geq 0$ is the capital per-capita at time $t \in \mathbb{N}$, where K_t is the stock of capital and L_t is the labour force. Let $\sigma(k_t) : \mathbb{R}_+ \to \mathbb{R}$ be the elasticity of substitution between production factors, that is, if $f(k_t) : \mathbb{R}_+ \to \mathbb{R}$ is the production function, then $\sigma(k_t)$ represents a measure of the ease with which capital and labour can be substituted in $f(k_t)$. More precisely $\sigma(k_t)$ represents the elasticity of output per-capita with respect to the marginal product of labour (see Hicks [33] and Robinson [65]). Notice that if $\phi(k_t)$ is continuous and twice differentiable, then $\sigma(k_t)$ is calculated as follows (see Sato and Hoffman [66]):

$$\sigma(k_t) = \frac{-f'(k_t)[f(k_t) - f'(k_t)k_t]}{f(k_t)f''(k_t)k_t}. \tag{1}$$

As an example, if $f(k_t)$ is of CES type, i.e. $f(k_t) = (1 + k_t^p)^{\frac{1}{p}}$, then $\sigma(k_t) = \frac{1}{1-p}$ $\forall k_t \geq 0$, that is the elasticity of substitution between production factors does not depend on the capital per-capita level. Differently, if $f(k_t)$ has a different form, then $\sigma(k_t)$ could no longer be constant. Consider, for instance, the following functions.

(*i*) The VES production function proposed by Revankar is given by:

$$f(k_t) = Ak_t^a[1 + bak_t]^{1-a}, \quad A > 0, \quad 0 < a < 1, \quad b \geq -1, \quad \frac{1}{k_t} \geq -b \tag{2}$$

hence

$$\sigma(k_t) = 1 + bk_t \tag{3}$$

(it has been considered in growth models such as in Brianzoni et al. [19], Cheban et al. [22], Karagiannis et al. [40] and Grassetti et al. [29]).

(*ii*) The SIGMOIDAL production function can be formalized as:

$$f(k_t) = \frac{\alpha k_t^p}{1 + \beta k_t^p}, \quad \alpha > 0, \quad \beta > 0, \quad p \geq 2$$

so that

$$\sigma(k_t) = 1 + \frac{\beta p k_t^p}{p(1 - \beta k_t^p) - (1 + \beta k_t^p)} \tag{4}$$

(this function has been used in growth models such as in Capasso et al. [20], Brianzoni et al. [16], Michetti [51] and Brianzoni et al. [17]).

In both cases (*i*) and (*ii*), σ is a function of k_t.
Consider now a discrete time growth model $k_{t+1} = \phi(k_t)$, $k_t \in \mathbb{R}_+$, describing the evolution of

capital per-capita k_t, $t \in \mathbb{N}$ (and consequently of output $y_t = f(k_t)$). For instance one can consider the Solow-Swan growth model given by

$$k_{t+1} = \phi(k_t) = \frac{1}{1+n}[(1-\delta)k + sf(k_t)]$$

where $n \geq 0$ is the constant population growth, $\delta \in (0,1)$ is the depreciation rate of capital and $s \in (0,1)$ is the constant saving rate (see [67], [53], [55] and [16]); or the more recent Bhöm and Kaas Solow-Swan growth model with differential savings given by

$$k_{t+1} = \phi(k_t) = \frac{1}{1+n}[(1-\delta)k_t + s_w(f(k_t) - k_t f'(k_t)) + s_r k_t f'(k_t)]$$

(see [38], [37], [56], [12], [18], [19], [51] and [17]) where s_w and s_r are respectively the savings rate of workers and shareholders.
In all these cases the one dimensional map $k_{t+1} = f(k_t)$ describing the evolution of capital per-capita over time depends on both the capital per-capita and the elasticity of substitution between production factors, i.e. $k_{t+1} = \phi(k_t, \sigma(k_t))$. Given function f, the main focus of economic growth studies is to investigate the long term dynamics produced by $\phi(k_t)$ for a given initial state $k_0 > 0$. By following Medio and Lines [49], we recall the definition of an attractor for a discrete time dynamic system.

Definition 2.1. *Let Ω be the state space and define $\omega(x)$ as the set of all ω-limit points of x for a map. A compact invariant subset of the state space $\Lambda \subset \Omega$ is said to be an* ***attractor*** *if*

- (i) *its basin of attraction, or stable set, $B(\Lambda) = \{x \in \Omega \,|\, \omega(x) \subset \Lambda\}$, has strictly positive Lebesgue measure;*
- (ii) *there is no strictly smaller closed set $\Lambda' \subset \Lambda$ so that $B(\Lambda')$ coincides with $B(\Lambda)$ up to a set of Lebesgue measure zero.*

In particular, if we consider the map $k_{t+1} = \phi(k_t)$, where the state space is given by $\mathbb{R}_+$, then the attractor Λ can be a fixed point (therefore $\Lambda = k^*$), or it can be a n-cycle (so that $\Lambda = c_n = \{k_1^*, k_2^*, \ldots, k_n^*\}$), or a more complex set. If $\lim_{t\to\infty} f^t(k_0) = +\infty \quad \forall k_0 \in I(k_0, r) \cap \mathbb{R}_+$, then we state $\Lambda = (\infty)$.

Consider $\Lambda = k^*$ and assume that $\sigma(k_t) = \sigma \quad \forall k_t$, that is the elasticity of substitution between production factors is constant. Then if the capital per capital level increases (decreases) when the elasticity of substitution increases, i.e. $\frac{\partial k^*(\sigma)}{\partial \sigma} > 0$ (resp. $<$), we can conclude that the elasticity of substitution between production factors positively (resp. negatively) affects the capital per-capita equilibrium value $\forall k_0 \in B(k^*)$. However, if Λ is not a fixed point, then a first question that arises is how to measure the attractor of f. Secondly, if different attractors coexist, each one with its own basin, a second question that arises is which attractor must be considered in order to establish the relation between σ and Λ. Finally, the most important question to be considered is how to inspect the relation between elasticity of substitution and long term growth dynamics when also σ depends on k_t. In what follows we will suggest a way to tackle these questions. Recall that the elasticity of substitution between inputs measure the ease in which capital can be substituted by labour in production. Therefore, during boom and bust periods, the measures we will define in the following can be used by governments to evaluate economic policies able to reallocate resource between inputs with the purpose of reducing costs while avoiding losses in production.

2.2 Definitions

In this section some definitions are given in order to *measure* the asymptotic states and the elasticity of substitution associated to an attractor in an economic growth model and explain their relation.

2.2.1 Attractors and equilibrium levels

Recall that the economic growth model is given by $k_{t+1} = \phi(k_t, \sigma(k_t))$, therefore - whichever the production function (whether a constant or a non-constant elasticity of substitution production function) - the capital per-capita at time $t+1$ is equal to $\phi(k_t, \sigma(k_t))$. The prevailing interest is to inspect the long term dynamics of the economic growth model, that is to investigate the structure of the attractor of map f in the long term for an economic meaningful initial capital per-capita level $k_0 > 0$.
When the attractor is a fixed point $\Lambda = k^*$, the measure of the long term capital per-capita level is basic and exact being, indeed, equal to the fixed point itself. However, when the attractor consists of a more complex set, as a cycle or a complex attractor, its measure is not so immediate. Therefore, we propose a method to measure the attractor of the dynamic system using a synthetic one-value index.

Definition 2.2. *Let $k_{t+1} = \phi(k_t, \sigma(k_t))$, $f : A \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a discrete time one-dimensional system describing the evolution of capital per-capita k_t, where $\sigma(k_t)$ is the elasticity of substitution depending on k_t and $t \in \mathbb{N}$.*

- *If the attractor is $\Lambda_* = k^*$, then the measure of capital per-capita level associated to the attractor is*
$$k_{\Lambda_*} = k^*.$$
- *If the attractor is $\Lambda_\infty = +\infty$, then the measure of capital per-capita level associated to the attractor is*
$$k_{\Lambda_\infty} = +\infty.$$
- *If the attractor is a periodic cylce $\Lambda_{c_n} = c_n = \{k_1^*, k_2^*, \dots, k_n^*\}$, then three measures of the capital per-capita level associated to the attractor can be given:*
  - *the maximum capital per-capita level associated to Λ_{c_n}*
  $$k_{M\Lambda_{c_n}} = max\,\{k_i^* \ : \ k_i^* \in \Lambda_{c_n}\};$$
  - *the minimum capital per-capita level associated to Λ_{c_n}*
  $$k_{m\Lambda_{c_n}} = min\,\{k_i^* \ : \ k_i^* \in \Lambda_{c_n}\};$$
  - *the average capital per-capita level associated to Λ_{c_n}*
  $$\bar{k}_{\Lambda_{c_n}} = \frac{1}{n}\sum_{i=1}^{n} k_i^*, \quad k_i^* \in \Lambda_{c_n}.$$

- *If the attractor is a complex set, then it can be described by the following set*

$$\Lambda_{c_x} = \{k_i \ : \ k_i = f^i(k_0),\, N-p \leq i \leq N,\, k_0 \in B(\Lambda_{c_x})\}\,.$$

  i.e. Λ_{c_x} *is given by the last* p $k-$*values obtained by iterating the system* N *times (where* $N \gg p$ *are conveniently chosen, sufficiently high natural numbers) and three measures of capital per-capita level can be associated to the attractor:*

  - *the maximum capital per-capita level associated to* Λ_{c_x}

  $$k_{M\Lambda_{c_x}} = max\,\{k_i \ : \ k_i \in \Lambda_{c_x}\};$$

  - *the minimum capital per-capita level associated to* Λ_{c_x}

  $$k_{m\Lambda_{c_x}} = min\,\{k_i \ : \ k_i \in \Lambda_{c_x}\};$$

  - *the average capital per-capita level associated to* Λ_{c_x}

  $$\bar{k}_{\Lambda_{c_x}} = \frac{1}{p}\sum_{i=N-p}^{N} k_i, \quad k_i \in \Lambda_{c_x}.$$

With the previous definitions we established a method to measure the long term capital per-capita equilibrium in the case in which the attractor is a fixed point or a cycle or a more complex set. In the same line, we now expound how to measure the elasticity of substitution by taking into account the long term dynamics of the growth model.

2.2.2 Attractors and elasticity of substitution

As for the long term capital per-capita equilibrium, a measurement issue arises when attempting to measure the non-constant elasticity of substitution associated to an attractor. Also when measuring the elasticity of substitution between production factors one face a hurdle when the dynamics of the economic growth model are analyzed and the elasticity of substitution is associated to an attractor that may be complex. Recalling definition 2.2, the following definition determines a method to measure the elasticity of substitution associated to an attractor in the case in which function σ depends on the capital per-capita level.

Definition 2.3. *Let* $k_{t+1} = \phi(k_t, \sigma(k_t))$ *be a discrete time one-dimensional system describing the evolution of capital per-capita and consider definition 2.2.*

- *If the attractor is a fixed point* k^**, the elasticity of substitution associated to the attractor is*

$$\sigma_{\Lambda_*} = \sigma(k_{\Lambda_*}) \quad (where\ k_{\Lambda_*} = k^*)\,;$$

- *If the attractor is* $\Lambda_\infty = +\infty$*, the elasticity of substitution associated to the attractor is*

$$\sigma_{\Lambda_\infty} = \lim_{k_t \to +\infty} \sigma(k_t)$$

  if such a limit exists.

- *If the attractor is a n-cycle, three measures of elasticity of substitution can be associated to the attractor:*
  - *the maximum elasticity of substitution associated to* Λ_{c_n}

$$\sigma_{M\Lambda_{c_n}} = max\,\{\sigma(k_i^*)\ :\ k_i^* \in \Lambda_{c_n}\};$$

  - *the minimum elasticity of substitution associated to* Λ_{c_n}

$$\sigma_{m\Lambda_{c_n}} = min\,\{\sigma(k_i^*)\ :\ k_i^* \in \Lambda_{c_n}\};$$

  - *the average elasticity of substitution associated to* Λ_{c_n}

$$\bar{\sigma}_{\Lambda_{c_n}} = \frac{1}{n}\sum_{i=1}^{n}\sigma(k_i^*), \quad k_i^* \in \Lambda_{c_n}.$$

- *If the attractor is a complex set, three measures of elasticity of substitution can be associated to the attractor:*
  - *the maximum elasticity of substitution associated to* Λ_{c_x}

$$\sigma_{M\Lambda_{c_x}} = max\,\{\sigma(k_i)\ :\ k_i \in \Lambda_{c_x}\};$$

  - *the minimum elasticity of substitution associated to* Λ_{c_x}

$$\sigma_{m\Lambda_{c_x}} = min\,\{\sigma(k_i)\ :\ k_i \in \Lambda_{c_x}\};$$

  - *the average elasticity of substitution associated to* Λ_{c_x}

$$\bar{\sigma}_{\Lambda_{c_x}} = \frac{1}{p}\sum_{i=N-p}^{N}\sigma(k_i), \quad k_i \in \Lambda_{c_x}.$$

Since the elasticity of substitution seems to be a determinant of economic growth, we fixed the fundamentals to compare how the elasticity of substitution affects the growth process when a VES instead of CES production function is considered.

2.2.3 Measures of elasticity of substitution and capital per-capital level on the attractors

One of the fundamental topics linked to the research on economic growth models is to inspect how the elasticity of substitution affects the capital per-capita equilibrium level. As seen, the elasticity of substitution between production factors is given by (12). Consider a CES production function, then $\sigma(k_t) = \sigma\ \forall k_t \geq 0$ that is the elasticity of substitution does not depend on k_t. Therefore, it is easy to verify what occurs to long term dynamics (and hence to economic growth) when varying the elasticity of substitution (se [42], [43] and [53]). What is the relation between

elasticity of substitution and the long term dynamics of economic growth models when a non-constant production function is considered?
We take as examples the elasticity of substitution as given in (3) and (4):

$$\sigma(k_t) = 1 + bk_t \quad \text{ and } \quad \sigma(k_t) = 1 + \frac{\beta p k_t^p}{p(1-\beta k_t^p) - (1+\beta k_t^p)}.$$

In both cases σ depends on k and some parameters of interest (b or p and β respectively). Given an economic growth model $\phi(k_t, \gamma)$ and an elasticity of substitution function $\sigma(k_t, \gamma)$ (where γ is the parameter of interest while the other parameters are fixed), the previous definitions can be used as follows.

- Once the attractor Λ is determined, the capital per-capita level associated to the attractor k_Λ can be obtained.
- Furthermore, the elasticity of substitution associated to the attractor $\sigma(\Lambda)$ can be computed $\forall \gamma \in I_\gamma \subseteq R$ (where I_γ is the set of values that γ can assume, according to the hypothesis of the model).
- Finally it is possible to verify if - when γ is moved - k_Λ and σ_Λ move in the same direction, so that k_Λ and σ_Λ are positively correlated, or if k_Λ and σ_Λ move in the opposite directions, so that k_Λ and σ_Λ are negatively correlated.

2.3 Application on Kaldor model

Thanks to the definitions given above, we can now analyze the relation between the elasticity of substitution between production factors, long term dynamics and economic growth when a production function with non-constant elasticity of substitution is taken into account. In this section we present an applied example. In Brianzoni et al. [19] the dynamics of Kaldor's growth model with VES production function have been studied. The authors found all the attractors of the model and demonstrated that complex dynamics may arise if the elasticity of substitution is positive and lower than 1. However, they did not investigate the correlation between elasticity of substitution and capital per-capital equilibrium levels nor the implication on economic growth.

2.3.1 Existence and stability of attractors

In this section we briefly recall both the economic setup and outcomes achieved by Brianzoni et al. [19], in order to better explain how the new procedure herewith proposed can be applied. Consider Kaldor's [37] model, where workers and shareholders have different but constant saving rates. The one-dimensional map describing the evolution of capital per-capita is given by

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + s_w w(k_t) + s_r k_t f'(k_t)]$$

where $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are respectively the constant savings rates for workers and shareholder, $n > 0$ is the constant population growth rate and $t \in \mathbb{N}$. Wage rate $w(k_t)$ equals the marginal product of labour while the total capital income per worker of a shareholder is given by $k_t f'(k_t)$. Furthermore, consider the Revankar [62] production function, that is a VES production function, given by (2), where $A > 0$, $0 < a < 1$, $b \geq -1$ and $\frac{1}{k_t} \geq -b$.
When $b > 0$ the final growth model describing the capital per-capita evolution is given by

$$H(k_t) = \frac{1}{1+n}\left\{(1-\delta)k_t + A\left(\frac{k_t}{1+abk_t}\right)^a [s_w(1-a) + s_r(a+abk_t)]\right\} \tag{5}$$

which is strictly increasing with respect to k_t. Differently, when $-1 \leq b < 0$ the final growth model is given by

$$\phi(k_t) = \begin{cases} H(k_t) & \forall k_t \in [0, -\frac{1}{b}] \\ H(-\frac{1}{b}) & \forall k_t > -\frac{1}{b} \end{cases} \tag{6}$$

that is a continuous and piecewise smooth map, nonlinear for $k_t \in [0, -\frac{1}{b}]$ and with a flat branch for $k_t > -\frac{1}{b}$. Furthermore the elasticity of substitution between production factors is defined in (3). In what follows we briefly recall the main results reached in Brianzoni et al. [19].

Remark 2.4. *Consider the economic growth model be given by H or F defined in (5) and (6).*

(1) If $b > 0$, then:

- *(i) when $\frac{n+\delta}{A} > (ab)^{-a} a s_r b$, H has one stable fixed point given by $k_t = k^* > 0$ and one unstable fixed point given by $k_t = 0$;*
- *(ii) when $0 < \frac{n+\delta}{A} < (ab)^{-a} a s_r b$, H has a unique, unstable, fixed point given by $k_t = 0$.*

(2) If $b \in [-1, 0)$, F has a non-differentiable point given by $P = \left(-\frac{1}{b}, F(-\frac{1}{b})\right)$. Given $M = \left(-\frac{s_r}{b}\right)^{a-1} \frac{s_w^2(1-a) + a s_r(s_w - s_r)}{(s_w - a s_r)^a}$ and $N = s_w[-b(1-a)]^{1-a}$, then:

- *(a) Consider $s_r < s_w$,*
  - *(i) if $\frac{n+\delta}{A} \in [0, M)$, F has one unstable fixed point given by $k = 0$ and one superstable fixed point given by $k^* = F(-\frac{1}{b})$;*
  - *(ii) if $\frac{n+\delta}{A} \in (M, N)$, F has two unstable fixed points given by $k = 0$ and $k_2^* \in \left(-\frac{s_r}{bs_w}, -\frac{1}{b}\right)$, one locally stable fixed point given by $k_1^* \in \left(0, -\frac{s_r}{bs_w}\right)$ and one superstable fixed point given by $k^* = F(-\frac{1}{b})$;*
  - *(iii) if $\frac{n+\delta}{A} > N$, F has one unstable fixed point given by $k = 0$ and one locally stable fixed point given by $k_1^* \in \left(0, -\frac{s_r}{bs_w}\right)$.*
- *(b) Consider $s_r \geq s_w$*
  - *(i) if $\frac{n+\delta}{A} \in [0, N)$, F has one unstable fixed point given by $k = 0$ and one superstable fixed point given by $k^* = F(-\frac{1}{b})$;*
  - *(ii) if $\frac{n+\delta}{A} > N$, F has one unstable fixed point given by $k = 0$ and one positive fixed point given by $k^* \in \left(0, -\frac{1}{b}\right)$. Moreover*

· *if* $-\frac{1}{b} < F\left(-\frac{1}{b}\right)$, k^* *is superstable;*

· *if* $-\frac{1}{b} > F\left(-\frac{1}{b}\right)$, k^* *may be unstable and complex dynamics may be exhibited.*

Now that we have recalled the results obtained by Brianzoni et al. [19] on local and global dynamics, we can move on to our study to verify how the elasticity of substitution affects Kaldor's growth model with VES production function.

2.3.2 Elasticity of substitution associated to the attractors. Measures

In this section we want to establish if, when complex attractors emerge, the capital per-capita equilibrium level and the elasticity of substitution associated to the attractor are positively correlated, negatively correlated or not correlated. From Brianzoni et al. [19] we know that, when $b \in [-1, 0)$ and $s_r \geq s_w$, if $\frac{n+\delta}{A} > N$ and $-\frac{1}{b} > F\left(-\frac{1}{b}\right)$, then the positive fixed point may lose stability, so the economy fluctuates and cycles or more complex dynamics can be exhibited.

In order to analyze the attractors of map F while moving b, we proceed by making use of numerical experiments. Therefore, we fix all the parameters in F except for parameter b by considering the values available from international economic databases. For parameters s (saving rate) and n (exogenous labour growth rate) we consider the values given by OECD[1], Eurostat[2] and World Bank[3] for the annual saving rate from 1995 to 2014 (excluding negative values, as the growth model requires). For parameter A (total factor productivity index) we consider the estimate given by OECD[4] from 1993 to 2014. Therefore, the following table of admissible values for the parameters is assumed.

Parameter	Range value
Saving rate	$s \in [0.00015, 0.59]$
Labor growth rate	$n \in [0.00002, 0.27346]$
Total factor productivity	$A \in [56.8, 106]$

We fix $A = 57$, $n = 0.2$, $s_w = 0.01$ and $s_r = 0.05$. Moreover we set $a = 0.85$ and $b \in [-0.6, -0.2)$ in order to assure that conditions $(b.ii)$ of Remark 2.4 are satisfied. As it has been proved in Brianzoni et al. [19], in this case complex dynamics may occur, as we can observe in Figure 1.
We can now use the definitions given in section (2.2.1) to measure the attractors of the dynamic system. Figure 2 panel (a) shows the maximum capital per-capita level (blue line), the minimum capital per-capital level (green line) and the average capital per-capita level (red line) associated to the attractor.

[1]For saving rate see "OECD (2016), Saving rate (indicator). doi: 10.1787/ff2e64d4-en"; for labour growth rate see "OECD (2016), Labour force (indicator). doi: 10.1787/ef2e7159-en".

[2]For saving rate see "Eurostat, Household saving rate, code: tsdec240"; for labour growth rate see "Eurostat, LFS main indicators, code: lfsi-act-a".

[3]For saving rate see "Gross savings, World Bank national accounts data, Catalog Sources: World Development Indicators"; for labour growth rate see "Labor force, International Labour Organization using World Bank population estimates, Catalog Sources: World Development Indicators"

[4]See "OECD (2016), Multifactor productivity (indicator). doi: 10.1787/a40c5025-en".

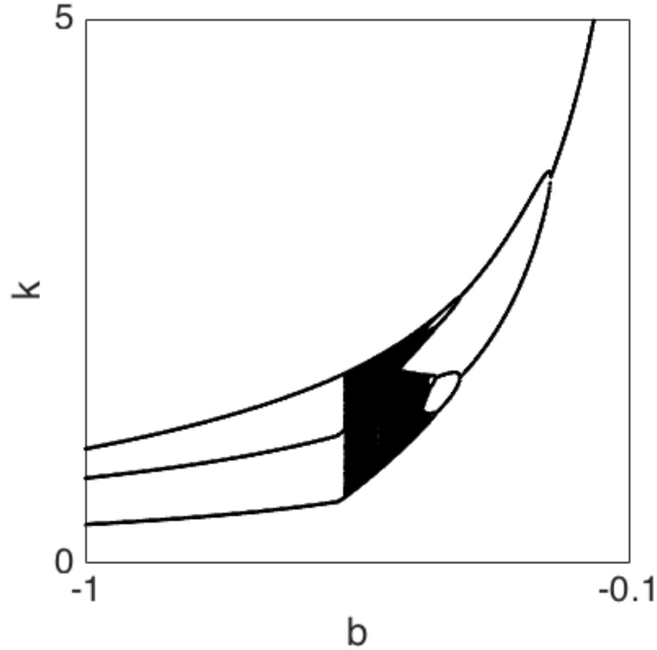


Figure 1: Bifurcation diagram w.r.t. b.

Thanks to the graphical analysis we can observe that the minimum capital per-capita level associated to the attractor increases as parameter b increases. For the maximum capital per-capita level associated to the attractor, a change in the curve slope can be observed in correspondence with the border collision bifurcation[5], as it can be seen in Figure 2 panel (b). In Figure 2 panel (d), the behaviour of the average capital per-capita level associated to the attractor can be observed: an increasing trend is visible in the sequence of the period doubling bifurcation cascade. We can conclude that, where a trend is visible, the capital per-capita level increases as the parameter associated to the elasticity of substitution is increased.
It is of importance to highlight that we are in a simplified state of definition (2.3), being the elasticity of substitution given by (3) a linear function; indeed the following proposition holds.

Proposition 2.5. *Let $\sigma(k_t)$ be a linear function of k_t and let the elasticity of substitution associated to an attractor σ_Λ be as defined in (2.3). Then:*

- $\sigma_{M\Lambda_{cn}} = \sigma(k_{M\Lambda_{cn}})$;
- $\sigma_{m\Lambda_{cn}} = \sigma(k_{m\Lambda_{cn}})$;
- $\bar{\sigma}_{\Lambda_{cn}} = \sigma(\bar{k}_{\Lambda_{cn}})$;
- $\sigma_{M\Lambda_{cx}} = \sigma(k_{M\Lambda_{cx}})$;

[5] About the border collision bifurcation occurring in the present model see Brianzoni et al. [19].

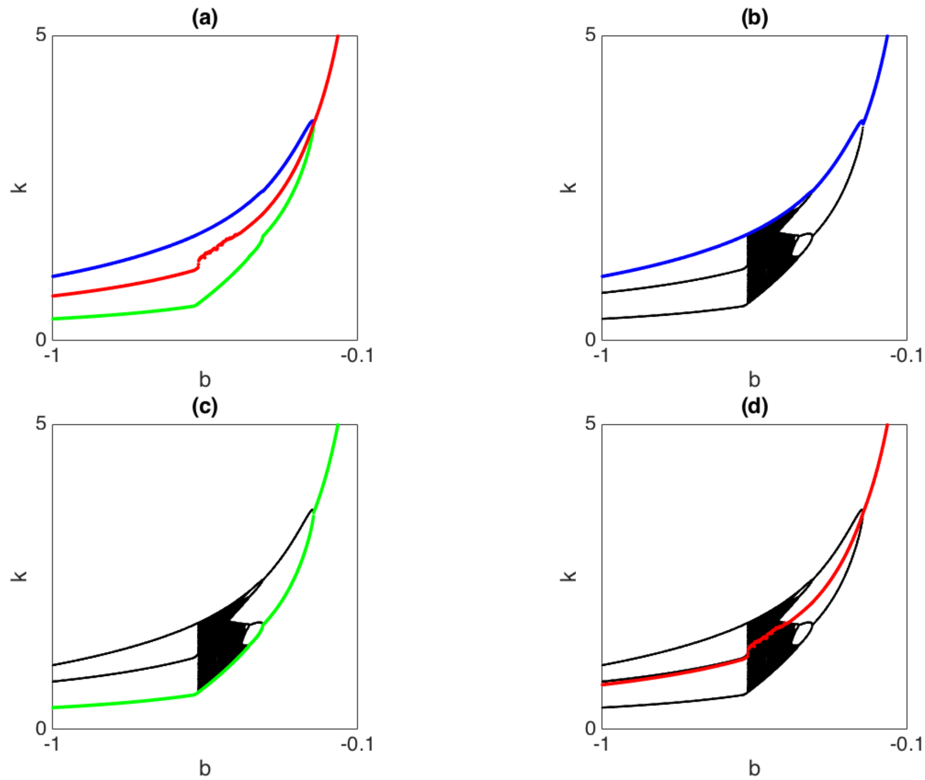


Figure 2: Bifurcation diagram w.r.t. b and maximum (blue), minimum (green) and average (red) capital per-capita level associated to the attractor.

- $\sigma_{m\Lambda_{cx}} = \sigma(k_{m\Lambda_{cx}})$;
- $\bar{\sigma}_{\Lambda_{cx}} = \sigma(\bar{k}_{\Lambda_{cx}})$.

Proof. Being the elasticity of substitution a linear function, for every set $K = \{k_1, k_2, \ldots k_n\}$ and $S = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ (with $\sigma_i = \sigma(k_i)$, $k_i \in K$),

- $max\{\sigma_i \in S\} = \sigma(k_M) | k_M \in K$ being $k_M \geq k_i \quad \forall k_i \in K$;
- $min\{\sigma_i \in S\} = \sigma(k_m) | k_m \in K$ being $k_m \leq k \quad \forall k_i \in K$;
- $\bar{\sigma} = \sigma(\bar{k})$, where $\bar{\sigma} = \frac{1}{n}\sum_{i=1}^{n} \sigma_i$ and $\bar{k} = \frac{1}{n}\sum_{i=1}^{n} k_i$.

□

In Figure 3 panel (a) the behaviour of the elasticity of substitution associated to the attractor can be observed.

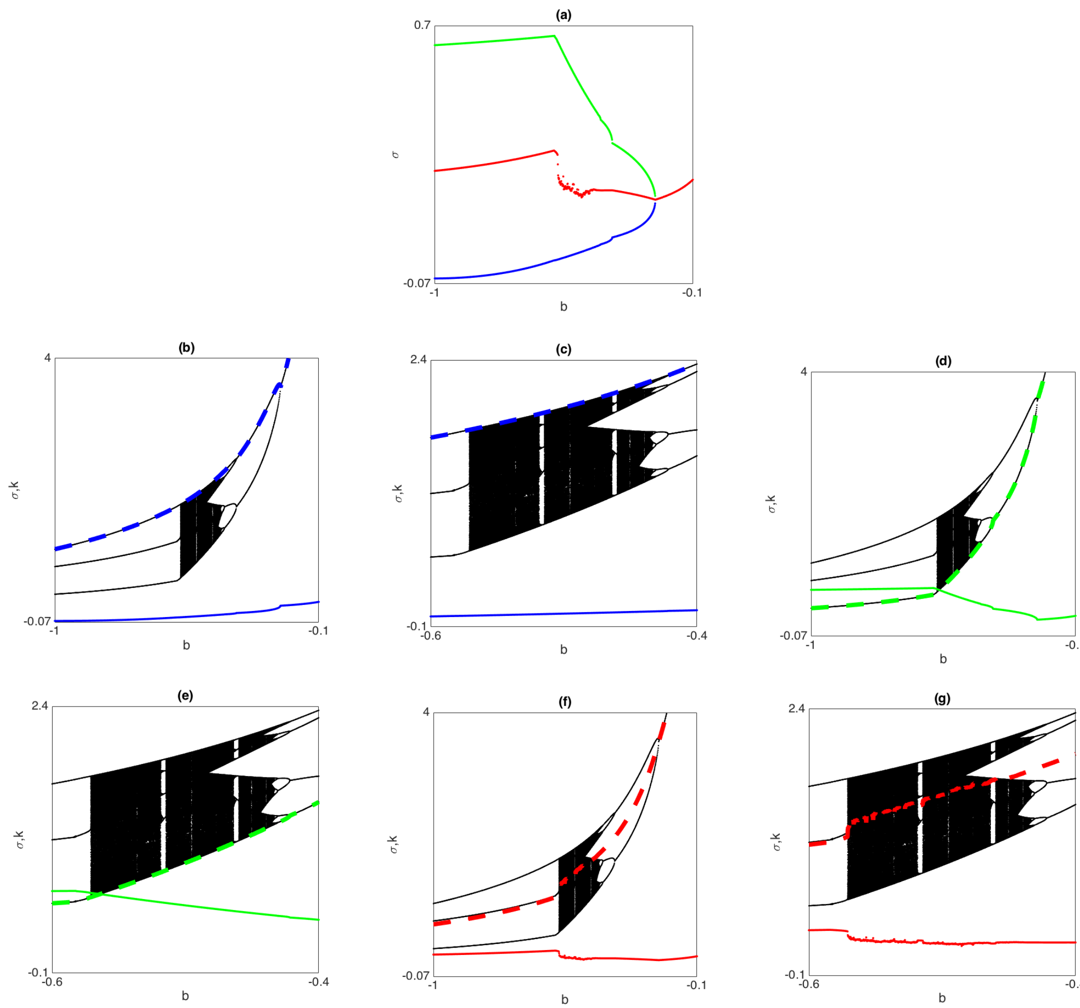


Figure 3: Maximum (blue), minimum (green) and average (red) elasticity of substitution and capital per-capita level (dashed) associated to the attractor.

Following the definitions given in proposition 2.5, we show the maximum elasticity of substitution associated to the attractor ($\sigma_{M\Lambda}$) (blue line), the minimum elasticity of substitution associated to the attractor ($\sigma_{m\Lambda}$) (green line) and the average elasticity of substitution associated to the attractor ($\bar{\sigma}_{\Lambda}$) (red line). We can observe the relation between the capital per-capita level and the elasticity of substitution associated to the attractor when moving parameter b. As far as the maximum capital per-capita level and its elasticity of substitution are concerned, in Figure 3 panel (b) it can be observed that both values increase when moving parameter b. A different behaviour is exhibited for the minimum capital per-capita level associated to the attractor and its elasticity of substitution, as it is shown in Figure 3 panel (d): a negative correlation between the two values is visible along the path until the border collision bifurcation occurs. Lastly, in Figure 3 panel (d), the relation between average capital per-capital level and the associated elasticity of substitution can be observed. Similarly to the minimum capital per-capita level a negative correlation is exhibited up to the border collision bifurcation. Notice that when the attractor is a fixed point, maximum, minimum and average capital per-capita level coincide and a positive correlation is shown.

To summarize, we disclose a numerical simulation in order to analyze the relation between the capital per-capita level associated to the attractor and the elasticity of substitution, when map F exhibits cycles and more complex dynamics. We observe that the maximum, minimum and average capital per-capita level associated to the attractor increase as the parameter b is increased.

Furthermore, we find a positive correlation between maximum capital per-capita level associated to the attractor and its elasticity of substitution. This result is similar to that obtained by Klump and La Grandville [42] using a CES production function in the Solow model: if two economies differ only for the level of their elasticity of substitution, the economy with the higher elasticity of substitution will have the higher capital per-capita level in the steady state. Conversely from Klump and La Grandville [42], a negative correlation between minimum and average capital per-capita level associated to the attractor and their elasticity of substitution is shown along the path.

3 Long run dynamics of Kaldor model with Shifted Cobb-Douglas technology

3.1 The economic setup

Consider the discrete time neoclassical one-sector growth model as proposed by Böhm and Kaas [12]: following Kaldor [38, 37] and Pasinetti [56] workers and shareholders have different but constant saving rates (respectively s_w and s_r), the labour force grows at rate n and δ is the depreciation rate of capital. Moreover, shareholders receive the marginal product of capital $f'(k)$ and the total capital income per worker is $kf'(k)$. We assume that the wage rate equals the marginal product of labor, that is

$$w(k) = f(k) - kf'(k). \tag{7}$$

Following Böhm and Kaas [12], the map describing the capital accumulation over time $t \in \mathbb{N}$ is given by

$$k_{t+1} = \phi(k_t) = \frac{1}{1+n}\left[(1-\delta)\,k_t + s_w w(k_t) + s_r k_t f'(k_t)\right], \quad k_t \geq 0 \tag{8}$$

where $n \geq 0$, $\delta \in (0,1]$, $s_w \in (0,1)$ and $s_r \in (0,1)$. Following Capasso et al. [20] we consider a Shifted Cobb-Douglas (SCD) production function that is a continuous non-concave and non-differentiable production function stating the existence of a critical level of capital needed before to get returns. This production function, differently from concave ones, well describes also non-developed countries since it takes in consideration the realistic need to establish a basic structure for production (as machineries and infrastructures) in order to obtain output. The SCD production function in its intensive form is given by

$$f(k_t) = \begin{cases} 0 & 0 \leq k_t \leq k_c \\ A(k_t - k_c)^{\alpha} & k_t > k_c \end{cases} \tag{9}$$

where $A > 0$ is the total productivity factor and $0 < \alpha < 1$ is the output elasticity of capital. $k_c \geq 0$ is the critical level of capital per-capita delimiting the poverty trap, that is the minimum capital per-capita initial level causing increasing returns since, when a country with almost no physical capital is considered, an initial investment is required before production (see Figure 4 (a)).

Notice that if $k_c \to 0^+$, then $f(k_t)$ approaches the Cobb-Douglas production function, therefore $f(k_t)$ can be considered as a generalization of the well-known Cobb-Douglas production function. Moreover

$$f'(k_t) = \begin{cases} 0 & 0 < k_t < k_c \\ \alpha A(k_t - k_c)^{\alpha-1} & k_t > k_c \end{cases}.$$

In order to assure non-negative wage and the framework having an economic meaning we assume that if $f(k) - kf'(k) < 0$ then the resulting wage is equal to zero, hence

$$w(k_t) = \begin{cases} 0 & 0 \leq k_t \leq k_w \\ A(k_t - k_c)^{\alpha-1}[(1-\alpha)k_t - k_c] & k_t > k_w \end{cases} \tag{10}$$

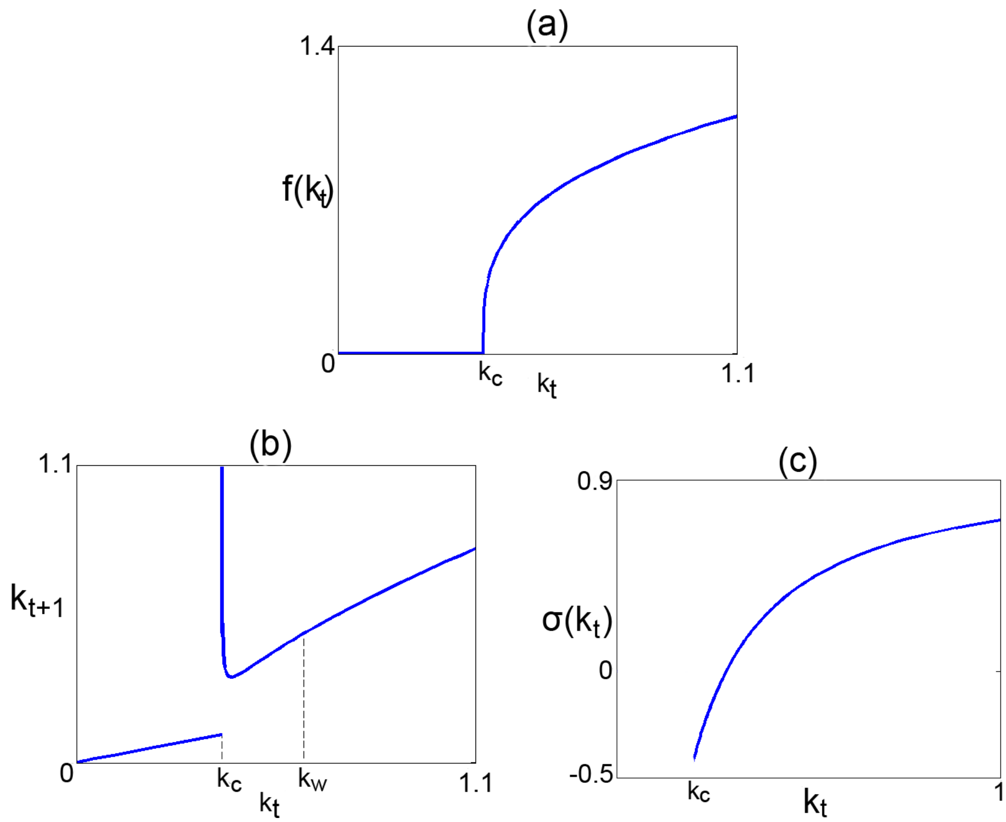


Figure 4: Common parameter values: $n = 0.3$, $\delta = 0.4$, $s_w = 0.6$ and $s_r = 0.7$. (a) SCD production function. Parameter values: $\alpha = 0.3$, $A = 1.2$ and $k_c = 0.4$. (b) The final map for capital accumulation. Parameter values: $\alpha = 0.5$, $A = 2.122$ and $k_c = 0.4$. (c) The elasticity of substitution. Parameter values: $\alpha = 0.3$, $A = 1.2$ and $k_c = 0.2$.

where $k_w = \frac{k_c}{1-\alpha} > k_c$. Taking into account equations (8), (9) and (10), the final one dimensional map describing the capital per-capita evolution is given by:

$$k_{t+1} = \phi(k_t) = \begin{cases} \frac{1-\delta}{1+n}k_t & 0 \le k_t \le k_c \\ \frac{1}{1+n}[(1-\delta)k_t + A\alpha s_r k_t (k_t - k_c)^{\alpha-1}] & k_c < k_t \le k_w \\ \frac{1}{1+n}\left\{(1-\delta)\,k_t + A\frac{s_w(k_t-k_c)+\alpha(s_r-s_w)k_t}{(k_t-k_c)^{1-\alpha}}\right\} & k_t > k_w \end{cases} . \tag{11}$$

We now discuss the main properties of map $\phi(k_t)$, i.e. the Solow-Swan growth model in the form given by Böhm and Kaas [12] with SCD as proposed by Capasso et al. [20]. Map ϕ is non negative, defined in $\mathbb{R}_+$ and it is discontinuous in $k_t = k_c$ since $\lim_{k_t\to k_c^-}\phi(k_t) = \frac{1-\delta}{1+n}k_c$ and $\lim_{k_t\to k_c^+}\phi(k_t) = +\infty$, while it is continuous in k_w being $\lim_{k_t\to k_w^-}\phi(k_t) = \lim_{k_t\to k_w^+}\phi(k_t) = \frac{1}{1+n}\left[\frac{1-\delta}{1-\alpha}k_c + A\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha}\right]$; furthermore $\lim_{k_t\to+\infty}\phi(k_t) = +\infty$ (see Figure 4 (b)). Observe that, for any fixed value of k_t, the capital per-capita level at time $t+1$ is always negatively influenced by the labour force growth rate (n) and the depreciation rate of capital (δ), whereas - passed the poverty trap i.e. $k_t < k_c$ - it is positively affected by the total productivity factor (A). Furthermore, for well developed countries, i.e. $k_t > k_c$ with k high enough, the higher the difference between workers and shareholders saving rates, the higher the capital per-capita at time $t+1$.

Recall that the elasticity of substitution between production factors for nonlinear production function is defined as follows (see Sato and Hoffman [66]):

$$\sigma(k_t) = -\frac{f'(k_t)[f(k_t) - f'(k_t)k_t]}{f(k_t)f''(k_t)k_t}\,. \tag{12}$$

and it is the measure of the ease in which capital and labour can be substituted in production. Being

$$f''(k_t) = A\alpha(\alpha-1)(k_t - k_c)^{\alpha-2} \qquad k_t > k_c$$

and being $\sigma = +\infty$ for linear production functions, the elasticity of substitution between production factors for the SCD can be easily calculated and it is given by

$$\sigma(k_t) = \begin{cases} +\infty & 0 < k_t < k_c \\ 1 - \frac{k_c}{(1-\alpha)k_t} & k_t > k_c \end{cases}.$$

Observe that $f(k_t)$ belongs to the class of Variable Elasticity of Substitution (VES) production functions, as $\sigma(k_t)$ depends on the level of capital per-capita k_t. Moreover $\sigma(k_t)$ is discontinuous in $k_t = k_c$ being $\lim_{k_t \to k_c^-} \sigma(k_t) = +\infty$ whereas $\lim_{k_t \to k_c^+} \sigma(k_t) = \frac{-\alpha}{1-\alpha} < 0$. Furthermore $\lim_{k_t \to +\infty} \sigma(k_t) = 1$. Notice that if $k_t > k_w > k_c$ then $\sigma(k_t) > 0$, whereas if $k_c < k_t < k_w$ then $\sigma(k_t) < 0$ (the graph of $\sigma(k_t)$ is in Figure 4 (c)). Notice also that σ is always lower then 1 for $k_t > k_c$. For what it concerns the sign of σ we observe that even if a negative elasticity of substitution between production factors is not conventional, several production functions in literature show negative elasticity of substitution (see Prywas [59], Andrikopoulos et al. [3], Thompson and Taylor [71], Nguyen and Streitwieser [54], Stern [68], Hamilton et al. [31] and Jurgen [35]), for instance, as suggested by Paterson [57], a negative elasticity of substitution can occur if complementary inputs are considered. Therefore, a negative elasticity of substitution between production factors for $k_c < k_t < k_w$ suggests that in the early stages of production, immediately outside the poverty trap, capital and labour are complementary and not replaceable.
As it has been discussed, map ϕ is defined in $\mathbb{R}_+$ and it is discontinuous in $k_t = k_c$. Map $\phi = 0$ iff $k_t = 0$, hence it passes through the origin. This is a trivial condition for economic growth models as no capital can be produced without capital. Moreover, map ϕ is always not negative. For $k_t \leq k_c$, map ϕ is a linear function passing through the origin with slope $m = \frac{1-\delta}{1+n}$. Note that m is positive and lower then 1, moreover it increases as δ or n decreases. Firstly we compute the derivative for map ϕ that is given by

$$\phi'(k_t) = \begin{cases} \frac{1-\delta}{1+n} & k_t < k_c \\ \frac{1}{1+n}\left\{1 - \delta + \alpha A\left[\frac{s_r(\alpha k_t - k_c)}{(k_t-k_c)^{2-\alpha}}\right]\right\} & k_c < k_t < k_w \\ \frac{1}{1+n}\left\{1 - \delta + \alpha A\left[\frac{s_r(\alpha k_t - k_c) + (1-\alpha)s_w k_t}{(k_t-k_c)^{2-\alpha}}\right]\right\} & k_t > k_w \end{cases}. \tag{13}$$

Notice that ϕ is non-differentiable in k_w and hence if an attractor A exists and $k_w \in A$ its stability must be discussed separately. For what it concerns the behaviours of map ϕ for sufficiently high levels of capital per capita we observe that $\forall k > k_c$ function ϕ may be strictly decreasing or it may present a turning point, i.e. a minimum point, as the following proposition states.

Proposition 3.1. *Let ϕ as given by (11). Assume $k_p = \frac{s_r k_c}{\alpha s_r + (1-\alpha)s_w}$ and $v = \frac{\delta - 1}{A}$. Then function ϕ is unimodal for $k_t > k_c$ with minimum point k_{min}.*

Proof. We define the function

$$G(k) = \begin{cases} s_r \alpha (k-k_c)^{\alpha-1} & k_c < k \leq k_w \\ \frac{s_w(k-k_c)+\alpha(s_r - s_w)k}{k(k-k_c)^{1-\alpha}} & k > k_w \end{cases} \tag{14}$$

where

$$G'(k) = \begin{cases} s_r \alpha (\alpha - 1)(k-k_c)^{\alpha-2} & k_c < k < k_w \\ \frac{(\alpha-1)[s_w + (s_r - s_w)\alpha]k^2 + (2-\alpha)s_w k_c k - s_w k_c^2}{k^2(k-k_c)^{2-\alpha}} & k > k_w \end{cases} \tag{15}$$

For $k > k_c$ function ϕ may be written in terms of function G defined in (14) as follows

$$\phi(k) = \frac{1}{1+n}\{(1-\delta)k + AkG(k)\} \qquad k > k_c \tag{16}$$

hence

$$\phi'(k) = \frac{1}{1+n}\{1-\delta + A[G(k) + kG'(k)]\} \qquad k > k_c, k \neq k_w \tag{17}$$

Then $\phi'(k) = 0$ iff $G(k) + kG'(k) = \frac{\delta-1}{A}$. Observe that $G(k) + kG'(k)$ can be written as follows:

$$H(k) = G(k) + kG'(k) = \begin{cases} s_r \alpha(\alpha k - k_c)(k-k_c)^{\alpha-2} & k_c < k < k_w \\ \frac{\alpha\{[s_w + (s_r - s_w)\alpha]k - s_r k_c\}}{(k-k_c)^{2-\alpha}} & k > k_w \end{cases}$$

hence the turning points of ϕ are solutions of

$$H(k) = \frac{\delta - 1}{A}. \tag{18}$$

Function $H(k)$ is such that $\lim_{k \to k_c^+} H(k) = -\infty$, moreover

$$H'(k) = \begin{cases} \frac{s_r\alpha(\alpha-1)(\alpha k - 2k_c)}{(k-k_c)^{3-\alpha}} & k_c < k < k_w \\ \frac{\alpha(1-\alpha)\{(2s_r - s_w)k_c - [(1-\alpha)s_w + \alpha s_r]k\}}{(k-k_c)^{3-\alpha}} & k > k_w \end{cases} .$$

Assume $k_p = \frac{s_r k_c}{\alpha s_r + (1-\alpha)s_w}$ and $z = s_r(2\alpha - 1)\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$.
We first consider solutions of equation (18) for $k \in (k_c, k_w)$:

(i) for $\alpha > \frac{1}{2}$, function $H(k) < 0$ iff $k_c < k < \frac{k_c}{\alpha}$, therefore $H(k)$ can intersect the constant and negative function $v = \frac{\delta - 1}{A}$ only in the interval $I_1 = \left(k_c, \frac{k_c}{\alpha}\right)$. Moreover $H'(k) > 0 \forall k \in I_1$ and $\lim_{k \to \frac{k_c}{\alpha}^-} H(k) = 0$. So that $H(k) = v$ has always one solution;

(ii) for $\alpha \leq \frac{1}{2}$, function $H(k) < 0 \forall k \in (k_c, k_w) = I_2$, $H'(k) > 0 \forall k \in I_2$ and $\lim_{k \to k_w} H(k) = z$, so that $H(k) = v$ has one solution in the interval I_2 if $v \leq z$;

We now consider solutions of equation (18) for $k > k_w$:

(iii) for $\alpha < \frac{s_r - s_w}{2s_r - s_w}$ and $s_r > s_w$, function $H(k) < 0$ for $k \in (k_w, k_p) = I_3$, moreover $\lim_{k \to k_w^+} H(k) = z + (1-\alpha)s_w \left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha - 1}$, $\lim_{k \to k_p^-} H(k) = 0$ and $H'(k) > 0 \forall k \in I_3$ so that $H(k) = v$ has one solution in the interval I_3 if $v \geq z + (1-\alpha)s_w \left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$.

Since $z < z + (1-\alpha)s_w \left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$ cases (ii) and (iii) can not occur simultaneously, and hence at most one turning point may exists for $\alpha < \frac{s_r - s_w}{2s_r - s_w}$. Notice that for $z < v < z + (1-\alpha)s_w \left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$ equation (18) has no solution. Nevertheless for these parameter values it has $\lim_{k \to k_w^-} H'(k) < 0$ and $\lim_{k \to k_w^+} H'(k) > 0$ and hence $\phi(k)$ is unimodal with minimum point $k_{min} = k_w$. $\square$

Note that if condition (i) holds then $k_{min} < \frac{k_c}{\alpha}$, if condition ($ii$) holds then $k_{min} \leq k_w$ while if condition (iii) holds then $k_{min} \in (k_w, k_p)$. In all other cases $k_{min} = k_w$.

3.2 Long run dynamics

In this section we consider the question of the existence of steady states of system (11) and then we discuss about the local stability.

3.2.1 Existence of equilibrium levels

The problem of finding the number of steady states is not trivial to solve, considering the high number of parameters. As a general result, the map ϕ always admits one fixed point characterized by zero capital per capita, i.e. $k = 0$ is a fixed point for any choice of parameter values. Anyway, steady states which are economically interesting are those characterized by positive capital per worker. As previously underlined ϕ is a discontinuous map. Moreover, no positive fixed point exists for $0 < k_t \leq k_c$, being $0 < \frac{1-\delta}{1+n} < 1$. In order to determine the positive fixed points of ϕ with $k_t > k_c$ we consider function G as given in (14). The positive steady states of map ϕ are the solutions of equation

$$G(k_t) = \frac{n + \delta}{A}. \tag{19}$$

The following proposition concerning the number of steady states of the Solow growth model with SCD and differential saving can be proved.

Proposition 3.2. *Let ϕ as given by (11).*
Define $g = \frac{n+\delta}{A}$ and $k_M = k_c \frac{(2-\alpha)s_w + \sqrt{\alpha s_w[(4s_r - 3s_w)\alpha - 4(s_r - s_w)]}}{2(1-\alpha)[s_w + (s_r - s_w)\alpha]}$.

(*i*) *Assume $s_r \geq s_w$. Then ϕ has two fixed points given by $k_t = 0$ and $k^* > k_c$. Moreover*

(*a*) *if $g \geq G(k_w)$, $k^* \leq k_w$;*

(*b*) *if $g < G(k_w)$, $k^* > k_w$.*

(*ii*) *Assume $s_r < s_w$. Then*

(*a*) *if $g > G(k_M)$ there exist two fixed points given by $k_t = 0$ and $k^* < k_w$;*

(*b*) *if $g = G(k_M)$ there exist three fixed points given by $k_t = 0$, $k_1 < k_w$ and $k_2 = k_M$;*

(*c*) *if $G(k_M) < g < G(k_w)$ there exist four fixed points given by $k_t = 0$, $k_1 \in (k_c, k_w)$, $k_2 \in (k_w, k_M)$ and $k_3 \in (k_M, +\infty)$;*

(*d*) *if $g = G(k_w)$ there exist three fixed points given by $k_t = 0$, $k_1 = k_w$ and $k_2 > k_M$;*

(*e*) *if $g < G(k_w)$ there exist two fixed points given by $k_t = 0$ and $k^* > k_M$.*

Proof. $k_t = 0$ is a solution of equation $k_t = \phi(k_t)$ for all parameter values hence it is a fixed point for all parameter values. Being $\frac{1-\delta}{1+n} < 1$, for all $0 < k_t \leq k_c$ map ϕ does not intercept the main diagonal. Function G is such that $G(k_t) > 0 \forall k_t > k_c$, furthermore $\lim_{k_t \to k_c^+} G(k) = +\infty$ while $\lim_{k_t \to \infty} G(k_t) = 0$. $G(k_t)$ is continuous in k_w being $\lim_{k_t \to k_w^-} G(k_t) = \lim_{k_t \to k_w^+} G(k_t) = G(k_w) = \alpha^\alpha s_r \left(\frac{k_c}{1-\alpha}\right)^{\alpha-1}$. Moreover $G'(k_t)$ is given in (15). We distinguish between the following cases.

(*i*) If $s_r \geq s_w$, $G(k_t)$ is strictly decreasing since $G'(k_t) \leq 0$ and $G'(k_t) = 0$ at most in one point. Hence $G(k_t)$ intersects the positive and constant function $g = \frac{n+\delta}{A}$ in a unique value $k^* > k_c$.

(*ii*) If $s_r < s_w$, $\exists k_M > k_w$ such that $G'(k_t) < 0$ for $k_c < k_t < k_w \vee k_t > k_M$ and $G'(k_t) > 0$ for $k_w < k_t < k_M$. The local minimum and maximum points of function G are given by k_w and $k_M = k_c \frac{(2-\alpha)s_w + \sqrt{\alpha s_w[(4s_r - 3s_w)\alpha - 4(s_r - s_w)]}}{2(1-\alpha)[s_w + (s_r - s_w)\alpha]}$ respectively. Hence, if $\frac{n+\delta}{A} > G(k_M)$ or $\frac{n+\delta}{A} < G(k_w)$, then equation $G(k_t)$ intersects the positive and constant function $g = \frac{n+\delta}{A}$ in a unique positive value $k_t = k^* > k_c$. Whereas, if $G(k_w) < \frac{n+\delta}{A} < G(k_M)$ then equation $G(k_t) = g$ admits three positive solutions k_1, k_2, k_3, where $k_1 \in (k_c, k_w)$, $k_2 \in (k_w, k_M)$, $k_3 > k_M$. For $g = G(k_w)$ a border collision bifurcation occurs with the merging of the fixed point with the kink point of ϕ while for $g = G(k_M)$ a fold bifurcation occurs since the constant function g is tangent to G in the maximum point and it intersects function G in a second point $k^* < k_w$.

□

Taking into account Proposition 3.2, the Solow growth model with differential saving and shifted Cobb-Douglas production function always admits the equilibrium $k = 0$ moreover multiple equilibria can exist: up to three positive fixed points are exhibited depending on the parameter values (see Figure 5). Note that the necessary condition for the existence of more then one positive equilibria is

$s_r < s_w$, moreover more than one positive fixed point can emerge for sufficiently high values of the output elasticity of capital α. Note that these results agree with those obtained by Brianzoni et al. [19] considering the Revankar ([63]) VES production function: up to three positive fixed points may emerge if the elasticity of substitution is lower than one and workers save more than shareolders. Differently, when a CES production function is considered, at most two positive fixed point may emerge (see Brianzoni et al. [14]).

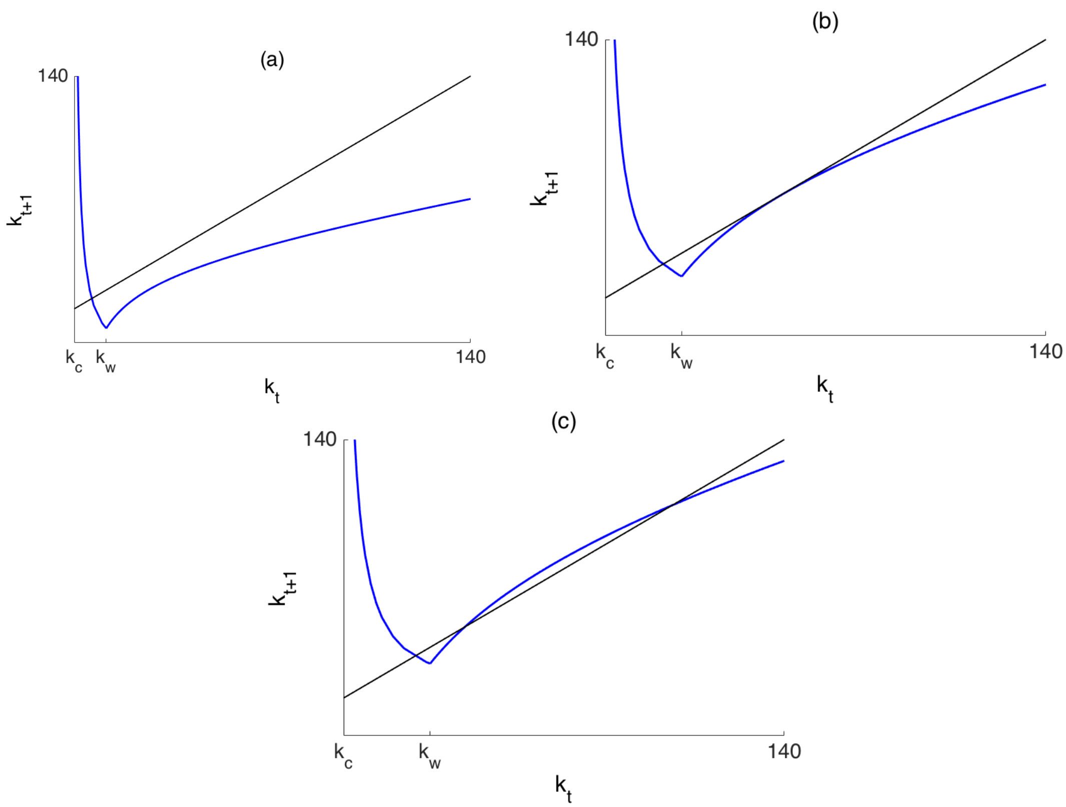


Figure 5: Map ϕ and its positive fixed points for $k_t > k_c$ in case of $s_r < s_w$ for the following parameter values: $\delta = 0.65$, $s_w = 0.45$, $s_r = 0.25$, $n = 0.45$, $A = 100$, $k_c = 44$. (a) One positive fixed point for $\alpha = 0.15$, (b) two positive fixed points for $\alpha = 0.275$, (c) three positive fixed points for $\alpha = 0.4$.

We want to highlight how the output elasticity of capital α and the difference between saving rates influence the number of steady states. For this purpose we define $\Delta_s = s_r - s_w$, $\Delta_s \in (-s_w, 1-s_w)$. Taking into account the conditions related to the existence and number of fixed points stated in Proposition 3.2, it is possible to describe how the number of fixed points varies as the output elasticity of capital α or the difference between saving rates Δ_s is moved. To the scope we fix all the parameter values but α and Δ_s and we consider several parameters combinations (Δ_s, α) taken on the set $\Omega = [-s_w, 1-s_w] \times [0,1]$. Define

$$C_1 = \{(\alpha, \Delta_s) \in \Omega : s_r - s_w = 0\} \tag{20}$$

$$C_2 = \left\{(\alpha, \Delta_s) \in \Omega : \frac{n+\delta}{A} - G(k_w) = 0\right\} \tag{21}$$

$$C_3 = \left\{(\alpha, \Delta_s) \in \Omega : G(k_M) - \frac{n+\delta}{A} = 0, s_r < s_w\right\} \tag{22}$$

then curves C_1, C_2 and C_3 separates the plane Ω into four regions: each region contains parameter values corresponding to a case stated in Proposition 3.2.

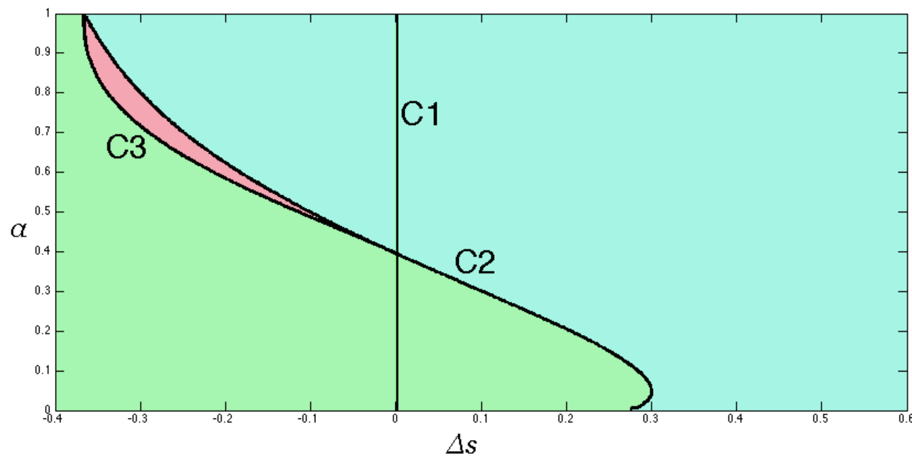


Figure 6: Parameter values: $\delta = 0.05$, $s_w = 0.4$, $n = 0.05$, $A = 3$, $k_c = 20$. Number of fixed points according to Proposition 3.2. In blue region 1 positive fixed point (cases ($i.b$) and ($ii.e$)), in green region 1 positive fixed point (cases ($i.a$) and ($ii.a$)), in red region 3 positive fixed points (case ($ii.c$)). Curves C_1, C_2 and C_3 are defined respectively in equations (20), (21) and (22).

The three regions are depicted in Figure 6: points on the right of C_1 curve verify the condition of the case (i) while the left region contain the parameter values related to the case (ii). The curve C_2 verifies the condition ($ii.d$) while the curve C_3 verifies the condition ($ii.b$). Notice that the existence of positive fixed points is due to high values of parameter α combined with low values of parameter Δ_s.

3.2.2 Stability of equilibrium levels

We now discuss about the local stability of the steady states of map ϕ. For what it concerns the local stability of the steady state $k = 0$, the following proposition holds.

Proposition 3.3. *Let ϕ as given by (11). Then the equilibrium $k = 0$ is always locally stable.*

Proof. Note that $\lim_{k_t \to 0^+} \phi'(k_t) = \frac{1-\delta}{1+n} \in [0,1)$ and consequently the origin is a locally stable fixed point for map ϕ. $\square$

Notice that for all initial conditions $k_0 \leq k_c$, map ϕ behaves as a contraction map and the iterations monotonically converge to $k = 0$. Therefore we define the poverty trap as a situation in which, at the initial time, the capital per capita level is not high enough, i.e. $k_0 \leq k_c$ and such that in the long term the economy will not survive. This result diverges from those obtained using a CES or VES production function (see Brianzoni et al. [14, 18, 19] and Grassetti et al. [29]) since a poverty trap exists (see also Capasso et al. [20] and Brianzoni et al. [16, 17]). Notice that CES and VES production functions well describe developed economies but are not able to capture the vicious circle of poverty that typically characterize non developed countries whereas the SCD production function allows to consider this phenomenon. Therefore, the presence of a poverty trap threaten the possibility of economic growth: economies starting from a low level of physical capital may be captured by the poverty trap and consequently the dynamic of physical capital will converge to zero. Note that for a small displacement from the stable equilibrium $k = 0$, the time trend of the relative displacement is $T_r = (\frac{1-\delta}{1+n})^t$. Therefore, if an economy lies in the poverty trap, an higher depreciation rate of capital or an higher labour force growth rate causes a faster return to the steady state characterized by zero capital per-capita.

As the long term dynamics produced by the model are completely known for all initial capital per per worker less then the threshold value k_c, we now focus on the growth patterns concerning sufficiently high initial states (i.e. $k_0 > k_c$).

For what it concerns the local stability of the positive hyperbolic steady state, the following proposition holds.

Proposition 3.4. *Let ϕ as given by (11) and recall Proposition 3.2.*

- (i) *Assume $s_r \geq s_w$. If $k^* > k_{min}$, the equilibrium k^* is locally stable. Otherwise $\phi'(k^*) < 0$.*
- (ii) *Assume $s_r < s_w$.*
  - (a) *Let $g > G(k_M)$. Then, if $k^* > k_{min}$ the equilibrium k^* is locally stable. Otherwise $\phi'(k^*) < 0$;*
  - (b) *Let $G(k_M) < g < G(k_w)$ then the fixed point $k_3 > k_M > k_{min}$ is always locally stable while the fixed point $k_w > k_2 > k_M$ is always unstable. Furthermore if $k_1 > k_{min}$, the equilibrium k_1 is always locally stable, whereas, if $k_1 < k_{min}$, then $\phi'(k_1) < 0$;*
  - (c) *Let $g < G(k_w)$. Then the equilibrium k^* is locally stable.*

Proof. Observe that $\phi'(k) = 1 + \frac{A}{1+n} k G'(k)$. Being $G'(k^*) < 0$ for cases (i), $(ii.a)$ and $(ii.e)$ of Proposition 3.2, then $\phi'(k^*) = 1 + \frac{A}{1+n} k^* G'(k^*) < 1$. Moreover ϕ is unimodal with minimum point k_{min}, so that if $k^* > k_{min}$ then $\phi'(k^*) \in (0, 1)$ whereas if $k^* < k_{min}$, $\phi'(k^*) < 0$. In case $(ii.c)$ k_1 is stable if $k_1 > k_{min}$ while $\phi'(k_1) < 0$ for $k_1 < k_{min}$. Moreover being $G'(k_2) > 0$ then $\phi'(k_2) > 1$ and consequently $k_3 > k_2 > k_{min}$ is locally stable being $\phi(k)$ strictly increasing $\forall k > k_{min}$. □

Notice that multiple equilibria coexist and hence multistability phenomena may occur. Therefore the global analysis of basins is mathematically significant (complex basins may exists) but especially economically relevant since it allow to answer one of the fundamental answers concerning developing countries and poverty trap: if an economy has a capital per capita level sufficiently high, i.e. $k_0 > k_c$, could it avoid the poverty trap?

Further considerations on the nature of the fixed points, their basins and their behaviour are debated in the following section.

3.3 Complex attractors

In this section we analyze the qualitative asymptotic properties of map ϕ using both numerical simulations and analytical tools. Note that the map may show complex dynamics if a fixed point is located on the decreasing branch (see Proposition 3.4). In order to consider the possibility of complex attractors to emerge we analyze the case in which $k_{min} > \phi(k_{min})$. Since the analytic form of function ϕ is complicated, we cannot analytically describe this condition and the dynamic behaviour has to be analyzed by numerical simulations.

Recall from Proposition 3.4 that, if a fixed point is placed in the interval (k_c, k_{min}) it may be locally stable or unstable and hence a more complex attractor A may appear. The following proposition states the existence of a trapping interval for map ϕ.

Proposition 3.5. *Let ϕ as given in (11). Assume $k_{min} > \phi(k_{min})$ and, if three positive fixed points exists, let $k_2 > \phi(k_{min})$. Then the set $J = [\phi(k_{min}), \phi^2(k_{min})]$ is trapping.*

Being ϕ unimodal it admits a trapping set J under the conditions of Proposition 3.5, moreover as J is trapping then if a complex attractor A exists, it must belong to it. Furthermore A must attract the trajectory starting from the turning point k_{min}. Recall that if $k_{min} > \phi(k_{min})$, then the eigenvalue of the fixed point placed on the decreasing branch of map ϕ is negative and hence it may lose stability only via period-doubling bifurcation. Notice that subsequent bifurcations may be of the border collision type.

In Figure 7 we show three different staircase diagrams of map ϕ with initial condition $k_0 = k_{min}$ and hence belonging to the trapping set J. In panel (a) a stable fixed point is presented for $\alpha = 0.5$. In panel (b) a stable cycle C_2 of period 2 is reached for $\alpha = 0.4$. Complexity emerges as the parameter α decreases and a complex attractor is visible in panel (c) for $\alpha = 0.35$. In order to discuss the bifurcations leading to chaos within the trapping interval J defined in Proposition 3.5, we take in consideration the role of the difference between saving propensities and the elasticity of substitution. Figure 8 (a) contains the sequence of bifurcation of map ϕ as the parameter Δ_s is moved while Figure 8 (b) shows the asymptotic dynamics versus the bifurcation parameter α.
In both the diagrams complex dynamics arise. Since $f(k_t)$ is a VES production function, we follow the method presented in Chapter 1 in order to measure both the *elasticity of substitution associated to the attractor* and the *capital per-capita associated to the attractor* in order to analyze the relation between elasticity of substitution and capital per-capita equilibrium levels. As Figure 9 shows, an higher capital per-capita equilibrium level is linked to an higher elasticity of substitution, confirm the results obtained by Klump and La Grandville [42] considering the Solow growth model with a normalized CES production function. Figure 10 presents a cycle cartogram showing a two-parametric bifurcation diagram qualitatively: each color represents a long-run dynamic behaviour for a given initial condition in the parameter plane (Δ_s, α). Cycles of different order are exhibited. Notice that if α is sufficiently small, then complex dynamics emerge if the difference between workers and shareholders is large enough. This result is in line with those obtained by Brianzoni et al.

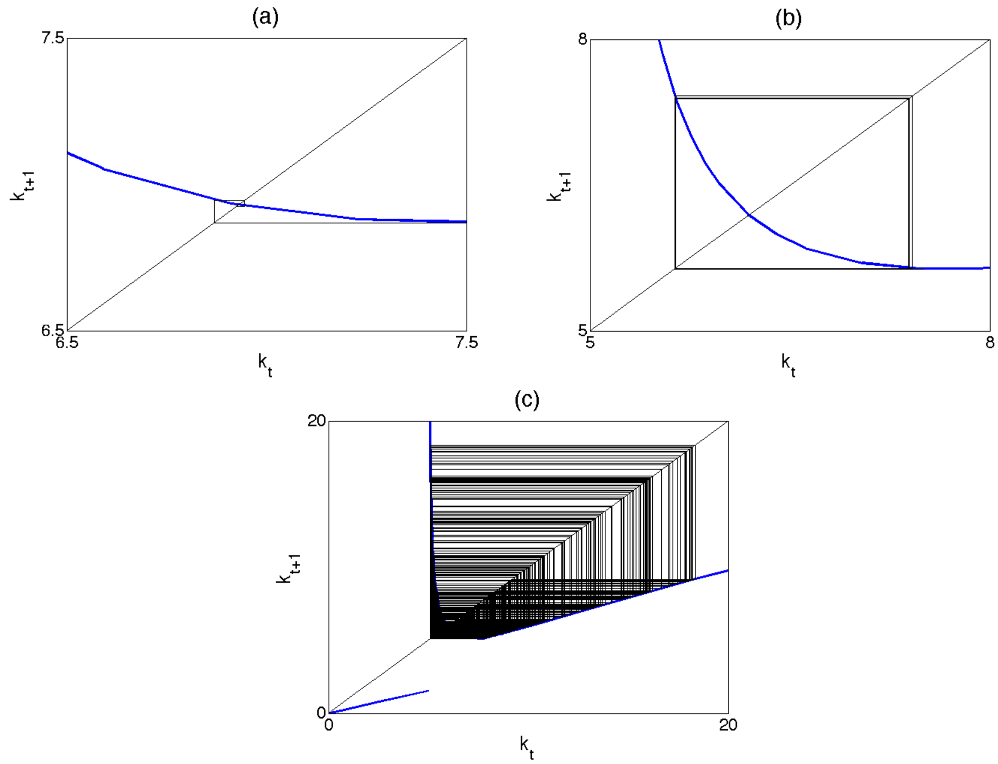


Figure 7: Staircase diagram of ϕ being $n = 0.3$, $\delta = 0.6$, $s_w = 0.25$, $s_r = 0.5$, $A = 5$, $k_c = 5$ and i.c. $k_0 = k_{min}$ for different values of α. (a) $\alpha = 0.5$, stable fixed point. (b) $\alpha = 0.4$, stable 2-period cycle. (c) $\alpha = 0.35$ complex attractor.

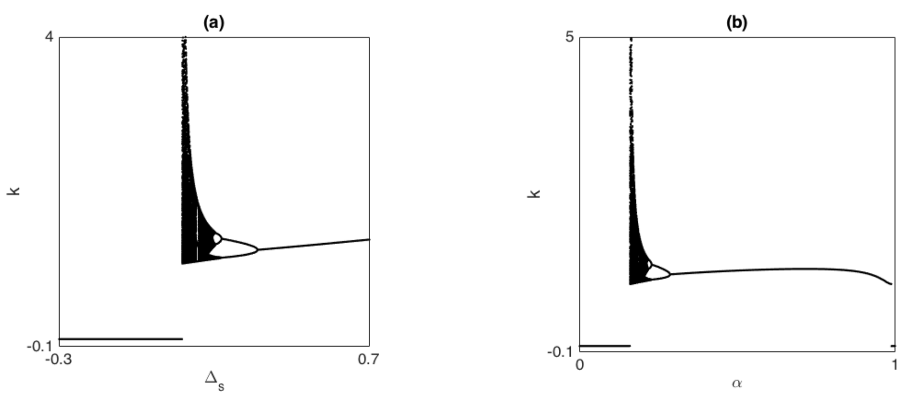


Figure 8: Parameter values $n = s_w = 0.3$, $\delta = 0.2$, $A = k_c = 1$ and $k_0 = k_{min}$. (a) Bifurcation diagram of ϕ w.r.t. Δ_s being $\alpha = 0.2$. (b) Bifurcation diagram of ϕ w.r.t. α being $s_r = 0.47$.

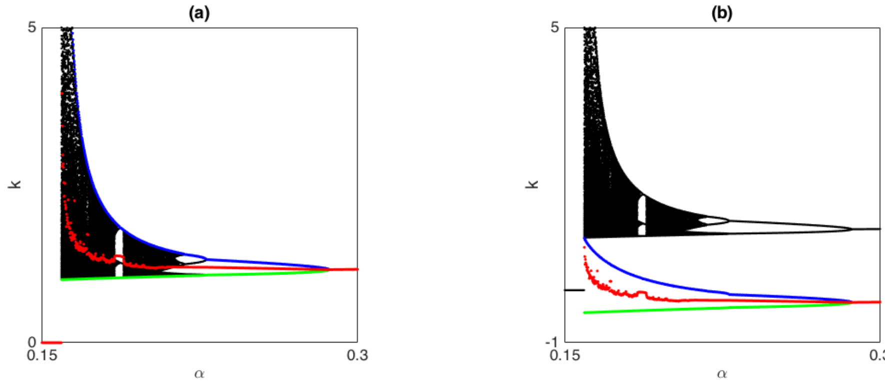


Figure 9: Capital per-capita associated to the attractor in panel (a) and elasticity of substitution associated to the attractor in panel (b) overlapping the bifurcation diagram of Figure 8 (b).

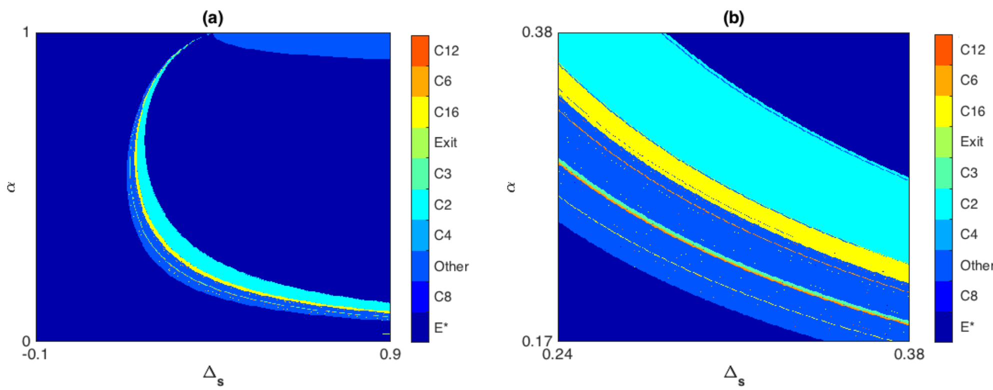


Figure 10: (a) Cycle cartogram of map ϕ in the plain (Δ_s, α) for $\delta = 0.2$, $n = 0.3$, $A = k_c = 1$, $s_w = 0.1$ and i.c. $k_0 = 1.05$. (b) An enlargement.

[14, 18, 19]. Note also that the elasticity of substitution increases as parameter α decreases and hence fluctuations arise when the elasticity of substitution between production factors is sufficiently high (but still lower then one).

Differently from previous literature, if the SCD production function is considered the Kaldor model well describes the long run dynamics of non developed, developing and developed countries. In particular it describes three different long run behaviour: convergence to the poverty trap, convergence to cycle or more complex set in which the economy alternates boom and bust periods and convergence to a positive capita per-capita value. Figure 11 shows a multistability phenomenon for map ϕ (about multistability see Bischi et al. [10], Brianzoni et al. [16, 17] and Sushko et al. [69]): given the same parameter values and two different initial conditions, in panel (a) a stable 2-period cycle is presented while panel (b) depicts the coexisting attractive fixed point.

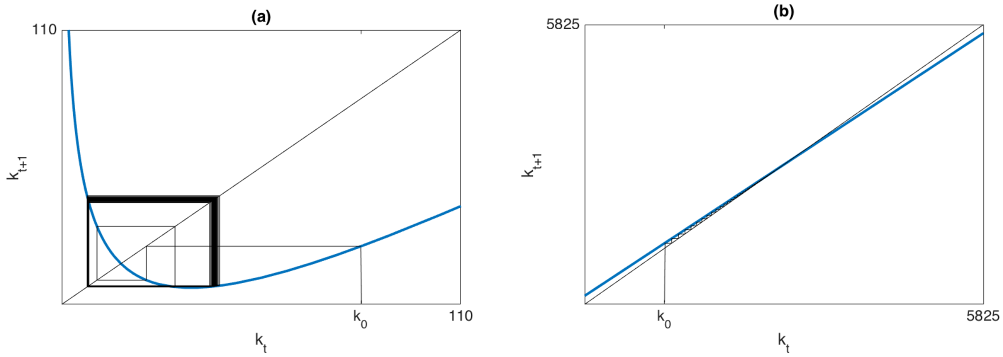


Figure 11: Staircase diagrams of ϕ for two coexisting attractors. Parameter values $\delta = 0.1$, $s_w = 0.9$, $s_r = 0.05$,$\alpha = 0.75$, $A = 14$, $k_c = 90$. (a) stable 2-period cycle. (b) stable fixed point.

3.4 Further developments

As further step the coexistence of attractors should be analyzed and compared with results obtained VES production function where if a multistability phenomenon appear, attractors are only positive fixed points.
Moreover several numerical experiments show that $\phi(k_{min}) > \phi(k_c)$ therefore $\phi(k_{min})$ and $\phi(k_c)$ separate the set $\mathbb{R}_+$ in three subset: $Z_1 = [0, k_c] \cup (k_{min})$, $Z_0 = (k_c, k_{min})$ and $Z_2 = (k_{min}, +\infty)$ whose points have respectively one, zero and two rank-1 pre-images so that ϕ is a $Z_1 - Z_0 - Z_2$ map. In this case complex basins may emerge. Their existence, structure and bifurcations as a parameter varies should be studied. The implications of a negative elasticity of substitution between inputs should be analysed.

4 Long run dynamics of Kaldor model with Kadiyala technology

4.1 The economic setup

Consider the Kaldor [37] growth model in which the capital intensity k at time $t+1$ is equal to the period-t capital intensity after depreciation plus investments made in period t, determined by the savings behaviours of workers and shareholders. Therefore the accumulation law can be written as

$$k_{t+1} = \frac{1}{1+n}((1-\delta)k_t + s(k_t)f(k_t)), \tag{23}$$

where $s(k_t)$ is the aggregate savings propensity, $\delta \in [0, 1]$ is the physical rate of capital depreciation and $n \geq 0$ is the exogenous labour force growth rate.
The total income of shareholders is the marginal product of capital $f'(k)$ so that the capital income per worker of a shareholder is $\pi(k) = kf'(k)$ while the per-capita wage of a worker equals the marginal product of labour $w(k) = f(k) - kf'(k)$.
Since a perfect competition on the capital market is assumed, the two income groups may have different savings propensities: respectively $s_w \in (0, 1)$ and $s_r \in (0, 1)$ for workers and shareholders. Then the total per-capita savings is $s_w w(k) + s_r \pi(k)$. Being $f(k) = w(k) + \pi(k)$, the aggregate savings propensity can be written as

$$s(k) = \frac{s_w w(k) + s_r \pi(k)}{f(k)}. \tag{24}$$

Notice that $s(k)$ is determined endogenously and depends on the income distribution and on the two savings propensities. Substituting $s(k)$ in equation (23) the time-one map describing capital accumulation can be written as

$$k_{t+1} = \frac{1}{1+n}((1-\delta)k_t + s_w w(k_t) + s_r \pi(k_t)) \qquad k_t \in \mathbb{R}_+. \tag{25}$$

Observe that for $s_r = s_w$ the aggregate savings propensities become constant and map (25) is reduced to the so called Solow-Swan growth model (see Swan [70] and Solow [67]) that preclude the possibility of fluctuation and complex dynamics, as previous literature demonstrates (see among all Dechert [24]).
In this work we consider Kadiyala [36] production function in which elasticity of substitution varies with the input ratio as one moves along the isoquant and it is symmetric to the end points 0 and ∞. Furthermore, Kadiyala production function is a generalization of CD and CES production functions and it includes the Lu-Fletcher [45] production function as a special case. Kadiyala [36] production function is given by

$$f(k_t) = A(ak_t^{2\rho} + 2bk_t^{\rho} + c)^{\frac{1}{2\rho}} \qquad k_t > 0, \tag{26}$$

where a, b and c are nonnegative, $a + 2b + c = 1$, $A > 0$, $\rho \leq \frac{1}{2}$, $\rho \neq 0$. Parameters a and c are share parameters, b is a reaction parameter between capital and labor, A stands for the neutral technical progress and ρ determines the degree of substitutability of the inputs. Notice that for $\rho < 0$, $f(k)$ is not defined in $k = 0$, moreover $\lim_{k \to 0} f(k) = 0$ so that function (26) can be extended to the origin in case of $\rho < 0$ by defining $f(0) = \lim_{k\to 0} f(k)$. The extended function is given by

$$\bar{f}(k_t) = \begin{cases} 0 & k = 0 \\ f(k_t) & k > 0. \end{cases} \tag{27}$$

Function (26) includes the CES [4] production function ($b = 0$) and the Lu-Fletcher [45] production function ($c = 0$). Moreover, $\bar{f}$ is bounded from below therefore capital is not essential in production as Figure 12 shows: every input combination along the isoquant produces the same output and each isoquant considers the combination in which $K = 0$.

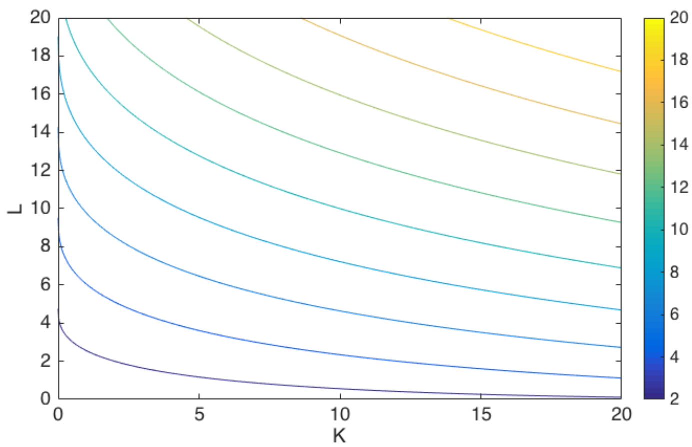


Figure 12: Isoquants for Kadiyala production function. Parameter values $A = 1$, $a = 0.1$, $b = 0.2$, $c = 0.5$ and $\rho = 0.4$.

Notice that

$$f'(k_t) = A\frac{k_t^{\rho-1}(ak_t^{\rho}+b)}{(ak_t^{2\rho}+2bk_t^{\rho}+c)^{1-\frac{1}{2\rho}}}, \tag{28}$$

and

$$f''(k_t) = Ak_t^{\rho-2}\frac{(\rho-1)abk_t^{2\rho}+[(2\rho-1)ac-b^2]k_t^{\rho}+(\rho-1)bc}{(ak_t^{2\rho}+2bk_t^{\rho}+c)^{2-\frac{1}{2\rho}}}, \tag{29}$$

hence $\bar{f}(k_t)$ is increasing and concave. Moreover, for $\rho > 0$, function (26) does not satisfy the Inada conditions being $\lim_{k_t\to\infty} f'(k_t) = Aa^{\frac{1}{2\rho}}$.
The elasticity of substitution between production factors is

$$\sigma(k_t) = \frac{1}{1-\rho+R}, \tag{30}$$

where $R = \frac{-\rho(ac-b^2)}{(ak_t^{-\rho}+b)(b+ck_t^{\rho})}$ and hence $\sigma \geq 1$ if and only if $\rho > 0$. Note that σ depends on k_t, which is why f belongs to the class of VES production functions. If $\rho > 0$, the maximum elasticity of substitution is

$$\sigma_{U^+} = \frac{1}{1-\rho-\rho\frac{ac-b^2}{\left(\sqrt{ac}+b\right)^2}}, \tag{31}$$

and it is reached at $k_t = \left(\frac{a}{c}\right)^{\frac{1}{2\rho}}$ while the minimum elasticity of substitution is

$$\sigma_{L^+} = \frac{1}{1-\rho}, \tag{32}$$

and it is reached at $k_t = 0$ and $k_t = +\infty$. If $\rho < 0$, the maximum elasticity of substitution is

$$\sigma_{U^+} = \frac{1}{1-\rho}, \tag{33}$$

and it is reached at $k_t = 0$ and $k_t = +\infty$ while the minimum elasticity of substitution is

$$\sigma_{L^+} = \frac{1}{1-\rho-\rho\frac{ac-b^2}{\left(\sqrt{ac}+b\right)^2}}, \tag{34}$$

and it is reached at $k_t = \left(\frac{a}{c}\right)^{\frac{1}{2\rho}}$. Therefore function (26) approaches the Cobb-Douglas production function as $\rho \to 0$, the Leontief production function as $\rho \to -\infty$ and the linear production function as $b \to 0$ and $\rho \to \frac{1}{2}$.
The capital income per worker of a shareholder is

$$\pi(k_t) = A\frac{k_t^\rho(ak_t^\rho + b)}{(ak_t^{2\rho} + 2bk_t^\rho + c)^{1-\frac{1}{2\rho}}}, \tag{35}$$

whereas the per-capita wage of a worker is

$$w(k_t) = A\frac{bk_t^\rho + c}{(ak_t^{2\rho} + 2bk_t^\rho + c)^{1-\frac{1}{2\rho}}}. \tag{36}$$

The final one-dimensional map describing the capital per capita evolution is given by

$$F(k_t) = \frac{1}{1+n}\left((1-\delta)k_t + A\frac{s_r ak_t^{2\rho} + (s_w+s_r)bk_t^\rho + s_w c}{\left(ak_t^{2\rho} + 2bk_t^\rho + c\right)^{1-\frac{1}{2\rho}}}\right) \qquad k_t \in \mathbb{R}_+. \tag{37}$$

Notice that for $\rho < 0$, $\lim_{k\to 0} F(k) = 0$ and $F(k)$ is not defined in 0. We hence assume a continuous extension $F(0) = 0$ for $\rho < 0$.

4.2 Boundedness of growth path

This section aims to analyze the quantitative shape of the capital accumulation map (37) taking into consideration the role of differential savings rates, in order to inspect their influence in the adjustments of an economy over time. For sake of convenience we denote the term

$$\alpha(k) := A\frac{s_r ak^{2\rho} + (s_w+s_r)bk^\rho + s_w c}{(ak^{2\rho} + 2bk^\rho + c)^{1-\frac{1}{2\rho}}}, \tag{38}$$

so that $F(k)$ can be written as $F(k) = \frac{1}{1+n}((1-\delta)k + \alpha(k))$.
We first consider the case in which $\rho < 0$. We introduce the following Lemma to analyze the quantitative shape of the capital accumulation F if the elasticity of substitution between production factors is lower than one.

Lemma 4.1. *The function* $\frac{\alpha(k)}{k}$ *is decreasing if* $0 < \rho < \frac{1}{2}$ *or if* $\rho < 0$ *and* $\min\{s_r, s_w\} > |2\rho(s_r - s_w)|$.

Proof. Since

$$\alpha'(k) = A\frac{2\rho s_r ak^{2\rho} + \rho(s_r + s_w)bk^\rho + (1-2\rho)(ak^{2\rho} + bk^\rho)\frac{s_r ak^{2\rho} + (s_w + s_r)bk^\rho + s_w c}{ak^{2\rho} + 2bk^\rho + c}}{k\left(ak^{2\rho} + 2bk^\rho + c\right)^{1-\frac{1}{2\rho}}},$$

it follows that $\frac{\alpha(k)}{k}$ is decreasing if and only if

$$\frac{\rho}{2\rho - 1}\left(\frac{s_r bk^\rho + s_w c}{bk^\rho + c} + s_w\right) < \frac{s_r ak^{2\rho} + (s_w + s_r)bk^\rho + s_w c}{ak^{2\rho} + 2bk^\rho + c}\,. \tag{39}$$

Let $0 < \rho < \frac{1}{2}$. Then the above inequality holds true for all parameters and all $k \in \mathbb{R}_+$.

Let $\rho < 0$, then the left-hand side of (39) is a decreasing function if and only if $s_r > s_w$ with

$$\lim_{k\to 0}\frac{\rho}{2\rho - 1}\left(\frac{s_r bk^\rho + s_w c}{bk^\rho + c} + s_w\right) = \frac{\rho}{2\rho - 1}(s_r + s_w)$$

and

$$\lim_{k\to\infty}\frac{\rho}{2\rho - 1}\left(\frac{s_r bk^\rho + s_w c}{bk^\rho + c} + s_w\right) = \frac{2\rho}{2\rho - 1}s_w\,.$$

The right-hand side of (39) is a monotonic function, moreover

$$\lim_{k\to 0}\frac{s_r ak^{2\rho} + (s_w + s_r)bk^\rho + s_w c}{ak^{2\rho} + 2bk^\rho + c} = s_r$$

and

$$\lim_{k\to\infty}\frac{s_r ak^{2\rho} + (s_w + s_r)bk^\rho + s_w c}{ak^{2\rho} + 2bk^\rho + c} = s_w.$$

Hence, two cases may occur. First, if $s_r < s_w$ then the left-hand side of (39) as well as the right-hand side of (39) are increasing. Thus, inequality (39) is satisfied if $s_r > \frac{2\rho}{2\rho-1}s_w$ or equivalently if

$$s_r > 2\rho(s_r - s_w).$$

However, following the standard assumption that shareholders safe more than workers that is $s_r > s_w$, it follows that inequality (39) is satisfied if $s_w > \frac{\rho}{2\rho-1}(s_r + s_w)$ or equivalently if

$$s_w > \rho(s_w - s_r).$$

Thus, if $\min\{s_r, s_w\} > |2\rho(s_r - s_w)|$, $\frac{\alpha(k)}{k}$ is decreasing. □

In the following Lemma it is shown that if the savings rate of the shareholders are sufficiently close to the savings rate of the workers, then map F is bounded from above by a linear capital accumulation equation independent from savings propensity of workers.

Lemma 4.2. *Let $\rho < 0$ and $\min\{s_r, s_w\} > |2\rho(s_r - s_w)|$. Then $F(k) \le F_r(k)$ $\forall k \in \mathbb{R}_+$, where*

$$F_r(k) := \frac{1}{1+n}\left((1-\delta)k + As_r a^{\frac{1}{2\rho}} k\right) .$$

Proof. Observe that $F_r(k) \ge F(k)$ if and only if

$$As_r a^{\frac{1}{2\rho}} \ge \frac{\alpha(k)}{k}. \tag{40}$$

Lemma 4.1 implies that $\frac{\alpha(k)}{k}$ is a decreasing function and moreover $\lim_{k\to 0} \frac{\alpha(k)}{k} = As_r a^{\frac{1}{2\rho}}$. □

Notice that previous Lemma holds if the elasticity of substitution between production factor is lower than one and hence the economy is characterized by a technology in which capital and labour are not easily substitutable. Then, if the difference between savings rates of the two income groups is sufficiently small the growth path of the economy is bounded from above by $F_r(k) = \frac{1}{1+n}\left((1-\delta)k + As_r a^{\frac{1}{2\rho}} k\right)$, as Figure 13 shows. F_r describes the growth behaviour of an economy in which technology is described by the linear production function $f_r(k) := Aa^{\frac{1}{2\rho}} k$, so that production is not possible without capital and all profits are given to shareholders. Map F_r has a unique steady state $k_r^* = 0$ if and only if $As_r a^{\frac{1}{2\rho}} \ne n+\delta$. Moreover, the steady state $k_r^* = 0$ is stable if and only if $s_r < \frac{n+\delta}{Aa^{\frac{1}{2\rho}}}$, so that an economy described by the accumulation law $F_r(k)$ would be captured from poverty trap if the savings rate of shareholders is not sufficiently high.
However, the assumption of savings rates of shareholders and workers which are sufficiently close to each other is rather strict and under this condition the presented model can be reduced to the Solow model with constant aggregate savings propensity. In the following Lemma it is shown that if the elasticity of substitution between inputs is lower than one, then the difference between saving behaviours plays a crucial role in the growth path.

Lemma 4.3. *Let $\rho < 0$. Assume $s_w > s_r$ or $s_w^{\frac{1}{2\rho}} > s_r > s_w$. Then $F(k) \le F_w(k)$ $\forall k \in \mathbb{R}_+$, where*

$$F_w(k) = \frac{1}{1+n}\left((1-\delta)k + As_w c^{\frac{1}{2\rho}}\right) .$$

Proof. Let $\rho < 0$. Observe that map F can be written as

$$F(k_t) = \frac{1}{1+n}\left((1-\delta)k_t + \alpha(k)\right) \tag{41}$$

and $\lim_{k\to\infty} \alpha(k) = As_w c^{\frac{1}{2\rho}}$. Thus, for sufficiently large $k \in \mathbb{R}_+$ the function $F(k)$ tends to the linear function $F_w(k)$. Moreover, $F(k) \le F_w(k)$ if and only if

$$\frac{s_r a k_t^{2\rho} + (s_w + s_r) b k_t^{\rho} + s_w c}{a k_t^{2\rho} + 2b k_t^{\rho} + c} \le \left(\frac{s_w c}{a k_t^{2\rho} + 2b k_t^{\rho} + c}\right)^{\frac{1}{2\rho}} . \tag{42}$$

The left-hand side of (42) is monotonic, moreover

$$\lim_{k\to 0} \frac{s_r a k^{2\rho} + (s_w + s_r) b k^{\rho} + s_w c}{a k^{2\rho} + 2b k^{\rho} + c} = s_r$$

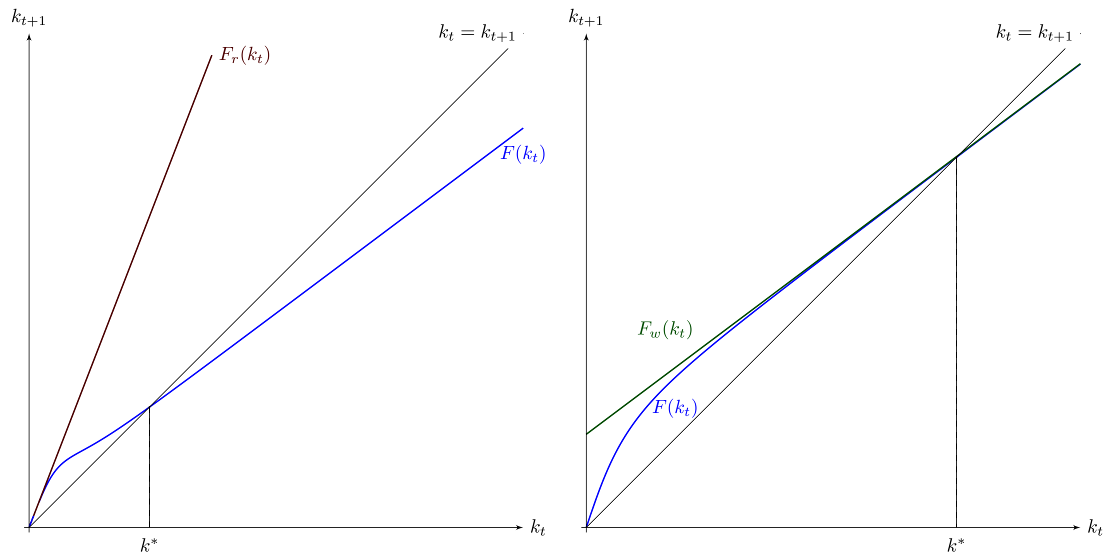


Figure 13: Map F and linear functions F_w and F_r. Parameter values $n = 0.2$, $\delta = 0.1$, $A = 3$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $\rho = -2$, $s_w = 0.6$, $s_r = 0.5$.

and

$$\lim_{k\to\infty} \frac{s_r a k^{2\rho} + (s_w + s_r) b k^{\rho} + s_w c}{a k^{2\rho} + 2bk^{\rho} + c} = s_w.$$

The right-hand side of (42) is a decreasing function with

$$\lim_{k\to 0} \left(\frac{s_w c}{a k_t^{2\rho} + 2b k_t^{\rho} + c} \right)^{\frac{1}{2\rho}} = \infty$$

and

$$\lim_{k\to \infty} \left(\frac{s_w c}{a k_t^{2\rho} + 2b k_t^{\rho} + c} \right)^{\frac{1}{2\rho}} = s_w^{\frac{1}{2\rho}}.$$

Since $s_w^{\frac{1}{2\rho}} > s_w$, it follows that $F(k) \leq F_w(k)$ if $s_r \leq s_w$. Otherwise if $s_r > s_w$, then $F(k) \leq F_w(k)$ if $s_w^{\frac{1}{2\rho}} > s_r$. $\square$

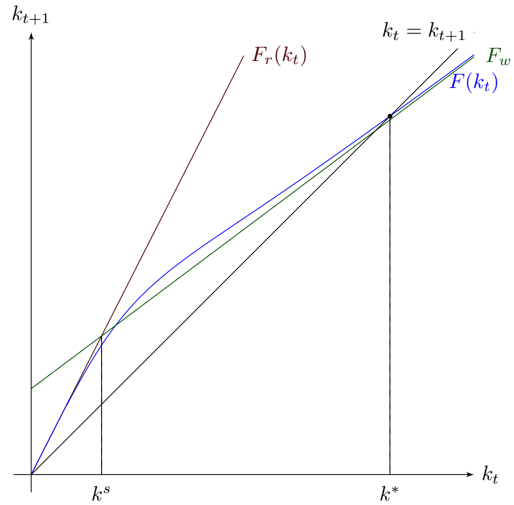
As Figure 13 shows, if workers save more than shareholders then the growth path is bounded from above by the linear function $F_w(k)$. Observe that $F_w(k)$ is independent of s_r so that the capital per-capita levels do not depend on the saving behaviours of shareholders. Notice also that $F_w(k)$ corresponds to the capital accumulation map in an economy with the constant production function $f_w(k) := A s_w c^{\frac{1}{2\rho}}$, so that capital is not necessary for production. It can easily be shown that $F_w(k)$ has a unique steady state $k_w^* = A \frac{s_w c^{\frac{1}{2\rho}}}{n+\delta}$, which is stable for all parameter values.

Lemma 4.2 and 4.3 imply that the capital per capita expansion can be bounded from above by the linear maps $F_r(k)$ and $F_w(k)$. Moreover, following Proposition holds.

Proposition 4.4. *Let $\rho < 0$. Then,*

- *if $F(k) \leq F_r(k)$ then $F(k) \leq F_w(k)$ for all $k \leq k^s = \frac{s_w}{s_r}\frac{c}{a}^{\frac{1}{2\rho}}$.*
- *if $F(k) \leq F_w(k)$ then $F(k) \leq F_r(k)$ for all $k \geq k^s = \frac{s_w}{s_r}\frac{c}{a}^{\frac{1}{2\rho}}$.*

Proof. Observe that $F_r(k) \leq F_w(k)$ if and only if $k \leq k^s = \frac{s_w}{s_r}\frac{c}{a}^{\frac{1}{2\rho}}$. Thus, if $F(k) \leq F_r(k)$ then $F(k) \leq F_w(k)$ for all $k \leq k^s$. Analogously, if $F(k) \leq F_w(k)$ then $F_w(k) \leq F_r(k)$ for all $k \geq k^s$ leading to $F(k) \leq F_r(k)$ for all $k \geq k^s$. □



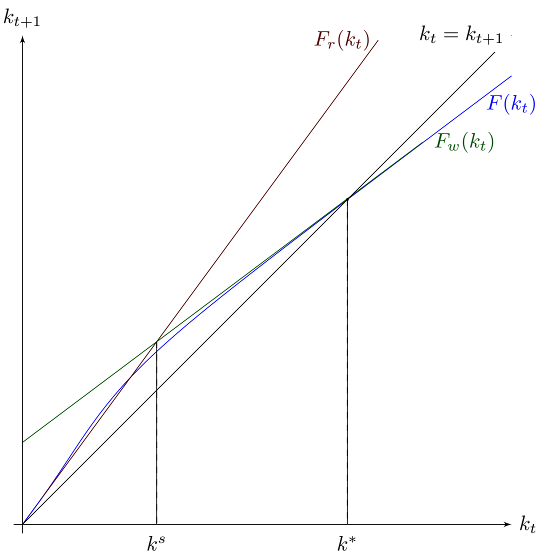


Figure 14: Map F and linear functions F_w and F_r. Parameter values $n = 0.2$, $\delta = 0.1$, $A = 3$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $\rho = -2$, $s_w = 0.6$, $s_r = 0.5$.

Notice also that $\lim_{k\to 0} F'(k) = \lim_{k\to 0} F_r'(k)$ and $\lim_{k\to+\infty} F'(k) = \lim_{k\to+\infty} F_w'(k)$ so that map F can be approximated by the linear function F_r for sufficiently small values of k and by the linear function F_w for sufficiently large values of the k, as numerous simulations showed (see Figure 14).

The economic meaning emerging from previous lemmas can be summarized as follows: assume capital and labour are not easily substitutable. Then the growth path of non-developed countries is influenced only by investments made by shareholders. In non-developed economies labour is unskilled and consequently wages are small so that the savings behaviour of workers can not influence growth path. In the early stage of growth investments of capitalist influence also the existence of poverty trap: small investment threaten the possibility of economic growth and a critical level of capital is needed in order to avoid the risk of fall into the "vicious circle of poverty". On the contrary, for developed economies the shape of growth path is influenced only by the savings behaviour of workers. Notice that results obtained can be used by policy-makers to increase the lower level an economy can reach during boom and bust periods and reduce fluctuations.

4.3 Long run dynamics

In this section we consider the question of the existence of steady states and then we discuss about the local stability.

4.3.1 Existence of equilibrium levels

In this section we consider the question of the existence and number of fixed points followed by a discussion on their stability. The establishment of the number of steady states is not trivial to solve, considering the high variety of parameters. We first consider the case in which $\rho > 0$ so that the elasticity of substitution between capital and labour is greater than one.

Lemma 4.5. *Let $\rho > 0$. Then $F(k)$ is an increasing function with $F(0) = \frac{A}{1+n} s_w c^{\frac{1}{2\rho}}$ and limiting slope $\lim_{k\to\infty} F'(k) = \frac{1}{1+n}\left(1 - \delta + A s_r a^{\frac{1}{2\rho}}\right)$.*

Proof. Consider the derivative of F, given by

$$F'(k) = \frac{1}{1+n}\left(1 - \delta + \alpha'(k)\right),$$

where

$$\alpha'(k) = A \frac{2\rho s_r a k^{2\rho} + \rho(s_r + s_w) b k^\rho + (1-2\rho)(ak^{2\rho} + bk^\rho)\frac{s_r a k^{2\rho} + (s_w + s_r) b k^\rho + s_w c}{ak^{2\rho} + 2bk^\rho + c}}{k\left(ak^{2\rho} + 2bk^\rho + c\right)^{1-\frac{1}{2\rho}}}$$

Hence, if $0 < \rho < \frac{1}{2}$, $\alpha'(k) > 0$ for all $k \in \mathbb{R}_+$ leading to $F'(k) > 0$. □

Lemma 4.5 implies that due to $F(0) = \frac{A}{1+n} s_w c^{\frac{1}{2\rho}} > 0$, the economy has at least one steady state if the limiting slope is smaller than one, that is if $s_r < \frac{n+\delta}{A} a^{-\frac{1}{2\rho}}$. In order to show that the fixed point is unique observe that the steady states of F are given by the solution of $\frac{\alpha(k)}{k} = n + \delta$, where $\alpha(k)$ is defined as in (38).

Proposition 4.6. *Let $\rho > 0$. Then F has a unique fixed point given by $k^* > 0$ if and only if $s_r > g$ where $g := a^{\frac{-1}{2\rho}} \frac{n+\delta}{A}$.*

Proof. As shown in Lemma 4.1 it follows that $\frac{\alpha(k)}{k}$ is decreasing if and only if

$$\frac{\rho}{2\rho - 1}\left(\frac{s_r b k^\rho + s_w c}{bk^\rho + c} + s_w\right) < \frac{s_r a k^{2\rho} + (s_w + s_r) b k^\rho + s_w c}{ak^{2\rho} + 2bk^\rho + c}. \tag{43}$$

If $0 < \rho < \frac{1}{2}$ the above inequality holds true for all parameters and all $k \in \mathbb{R}_+$. □

Notice that savings behaviour of shareholders influences the long-run dynamics of the economy: a threshold level s_r exists to entail ever-sustained growth. This result is consistent with that obtained by Brianzoni *et al.* [19] considering the Revankar [62] and the CES production functions. Moreover, as in Karagiannis *et al.* [40], when the elasticity of substitution is greater than one, then at most one positive fixed point can exist. Moreover, Proposition 4.6 implies that multistability phenomenon can not occur if $\rho > 0$. However, if the elasticity of substitution between capital and labour is smaller than one, multiple steady states may exist as shown in the following proposition.

Proposition 4.7. *Let F be given as in (37) and $\rho < 0$ with $g = a^{\frac{-1}{2\rho}} \frac{n+\delta}{A}$ as before. Then,*

(i) if $s_r \geq g$, F has at least two fixed points given by $k^* = 0$ and $k_1^* > 0$. Moreover, if $\min\{s_r, s_w\} > |2\rho(s_r - s_w)|$, the positive steady state is unique;

(ii) if $s_r < g$ and $\min\{s_r, s_w\} > |2\rho(s_r - s_w)|$, F has one fixed point given by $k^* = 0$.

Proof. Let $\rho < 0$, then $k = 0$ is always a solution for $F(k) = k$. Moreover, $\lim_{k\to 0} \frac{\alpha(k)}{k} = A s_r a^{\frac{1}{2\rho}}$ whereas $\lim_{k\to+\infty} \frac{\alpha(k)}{k} = 0$ and hence function $\frac{\alpha(k)}{k}$ intersects the positive constant function $\gamma = n + \delta$ at least once if $s_r \geq g$. Moreover, for $\min\{s_r, s_w\} > |2\rho(s_r - s_w)|$ Lemma 4.1 implies that $\frac{\alpha(k)}{k}$ is decreasing and hence the intersection point is unique. □

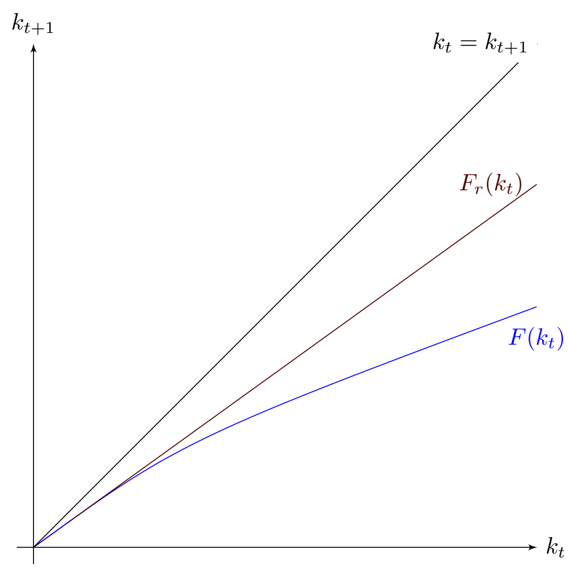


Figure 15: No positive fixed point for $n = 0.2$, $\delta = 0.56$, $A = 2.9$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $s_w = 0.12$, $s_r = 0.1$, $\rho = -3$.

Notice that if the elasticity of substitution between production factors is lower than one and the propensity to save of shareholders is sufficiently high, then multistability phenomenon may occur. In the following proposition it is shown that if multiple steady states occur, then the number of positive steady states is either two or four.

Proposition 4.8. *Let $\rho < 0$ and $s_r < g$. Then F has either zero, two or four positive steady states.*

Proof. Proposition 4.7 (ii) shows that if $\frac{\alpha(k)}{k}$ is decreasing then F has no steady states. Let $\frac{\alpha(k)}{k}$ be non-decreasing. Since $\lim_{k\to 0} \frac{\alpha(k)}{k} = As_r a^{\frac{1}{2\rho}}$ whereas $\lim_{k\to +\infty} \frac{\alpha(k)}{k} = 0$ it follows that if $s_r < g$ the number of intersection points of the constant line $\gamma = n + \delta$ with $\frac{\alpha(k)}{k}$ can be only an even number. Using the Bachmann - Landau notations, it follows that $\frac{\alpha(k)}{k} \in \mathcal{O}(k^4)$. Hence, the fundamental theorem of algebra implies that $\frac{\alpha(k)}{k} - (n + \delta)$ can have at most four zeros. This completes the proof. □

Further consideration about multiple steady states will be given in next section, using numerical simulation.

4.3.2 Stability of equilibrium levels

We now discuss about the stability of the steady states for positive values of ρ. The following proposition holds.

Proposition 4.9. *Let F be given in* (37) *and $\rho > 0$. Then the positive fixed point k^* is always globally stable.*

Proof. As shown by Lemma 4.5 $F'(k) > 0$. Thus, it is sufficient to show that $F'(k^*) < 1$ for the fixed point $k^* > 0$. This is the case if and only if $F'(k^*) = \frac{1}{1+n}(1 - \delta + \alpha'(k^*)) < 1$ or equivalently if $\alpha'(k^*) < n + \delta$. The positive fixed points k^* satisfy $\frac{\alpha(k^*)}{k^*} = n + \delta$, therefore

$$F'(k^*) < 1 \text{ if and only if } \left(\frac{\alpha(k^*)}{k^*}\right)' < 0.$$

Hence, the proof of Proposition 4.6 implies that for $\rho \geq 0$ it must be $F'(k^*) \in (0, 1)$. □

From the previous proposition it follows that for $\rho > 0$, the map F is strictly increasing and - if a positive fixed point exists - it is always globally stable. Recall that if $\rho > 0$ then $\sigma > 1$ so that, as in Brianzoni *et al.* [14, 18], when the elasticity of substitution between production factors is greater than one only simple dynamics can be produced. Notice that, for $\rho \to \frac{1}{2}$, f approaches the linear production function and hence when inputs are perfect substitutes no fluctuations may appear. Moreover, only the saving propensity of shareholders influences the existence of the steady state. This cases are resumed in Figure 16.
Notice that the difference between saving propensities influences the dynamics of the model for ρ negative, as the following propositions proves.

Proposition 4.10. *Let F be given in (37) and $\rho < 0$. If $s_r < \frac{1+n}{Aa^{\frac{1}{2\rho}}}$, the fixed point $k^* = 0$ is locally stable.*

Proof. $F(k)$ is non-differentiable in $k = 0$ so that behaviour of map F near the fixed point $k^* = 0$ must be considered. Since $\lim_{k\to 0} F'(k) = \frac{As_r a^{\frac{1}{2\rho}}}{1+n} > 0$ it follows that $\lim_{k\to 0} F'(k) < 1$ if and only if $s_r < \frac{1+n}{Aa^{\frac{1}{2\rho}}}$. □

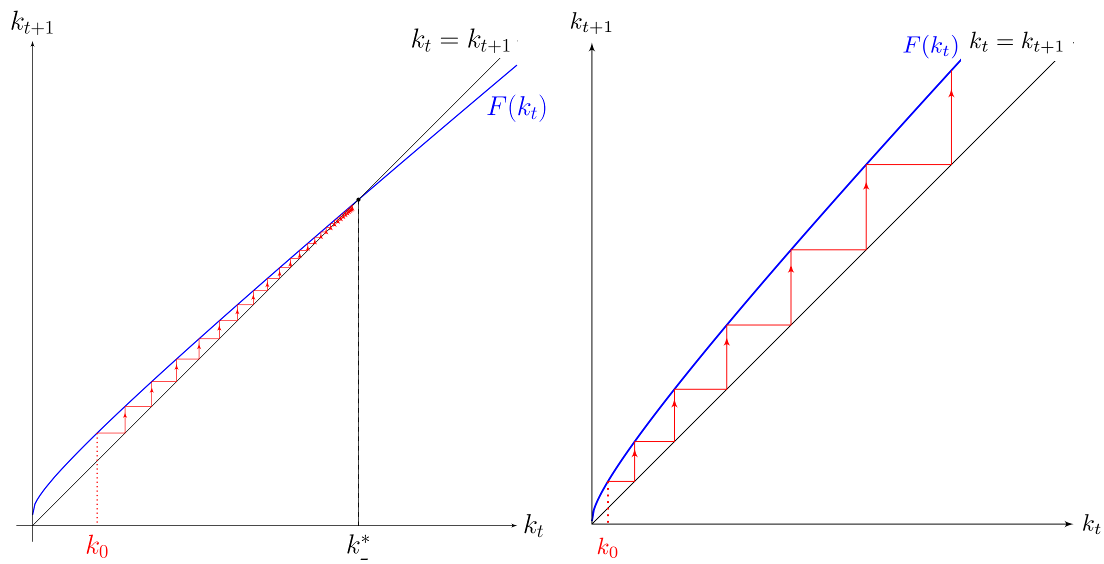


Figure 16: Map F for $\rho > 0$. Parameter values $n = 0.2$, $\delta = 0.1$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $\rho = 0.3$, $s_w = 0.4$, $s_r = 0.7$. (a) for $A = 0.9$ a positive fixed point exists and it is globally stable. (b) for $A = 10$ growth is unbounded.

Notice that, as in Capasso *et al.* [20] and Brianzoni *et al.* [16, 17] economies starting from a low level of physical capital may be captured by the poverty trap (see Figure 15). Moreover, differently from previous work we found that poverty trap can be avoided if the savings rate of shareholder is sufficiently large. This result is in line with that obtained by Chakraborty [21]: poverty traps may result if savings and investment rates are low, despite the absence of inefficient technology.

Proposition 4.11. *Let F be given in* (37) *and $\rho < 0$. If $s_w \geq s_r$, then for every fixed point k^* it follows that $F'(k^*) > 0$.*

Proof. As Böhm and Kaas [12] have shown, that equation (25) implies that $F'(k) \geq 0$ if and only if

$$1 - \delta + s_r f'(k) \geq (s_w - s_r) k f''(k)$$

and this inequality is satisfied for $s_w \geq s_r$. Hence, for $\rho < 0$ and $s_w \geq s_r$ it must be $F'(k^*) > 0$. This completes the proof. $\square$

Proposition 4.12. *Let F be given in* (37) *and $\rho < 0$. If $\max\{s_w, s_r\} < \frac{n+\delta}{Aa^{\frac{1}{2\rho}}}$, then for every fixed point k^* it follows that $F'(k^*) < 1$.*

Proof. As shown in the proof of Proposition 4.9 $F'(k^*) < 1$ if and only if $\left(\frac{\alpha(k^*)}{k^*}\right)' < 0$. Moreover, Lemma 4.1 implies that $\left(\frac{\alpha(k)}{k}\right)' < 0$ if and only if

$$\frac{\rho}{2\rho-1}\left(\frac{s_r b k^\rho + s_w c}{b k^\rho + c} + s_w\right) < \frac{s_r a k^{2\rho} + (s_w + s_r) b k^\rho + s_w c}{a k^{2\rho} + 2 b k^\rho + c}\,. \tag{44}$$

Substituting the right-hand side of (44) with $\alpha(k)$ and using the condition for a steady state it follows that $F'(k^*) < 1$ if and only if

$$\frac{\rho}{2\rho-1}\left(\frac{s_r b (k^*)^\rho + s_w c}{b (k^*)^\rho + c} + s_w\right) < \frac{n+\delta}{A} k^* \left(a (k^*)^{2\rho} + 2b (k^*)^\rho + c\right)^{\frac{-1}{2\rho}}\,. \tag{45}$$

The right-hand side of (45) is an increasing function with $\lim_{k\to 0} RHS(45) = \frac{n+\delta}{A} a^{\frac{-1}{2\rho}}$ and $\lim_{k\to\infty} RHS(45) = \infty$. Using the same argument as in Lemma 4.1 two cases may occur. First, if $s_r > s_w$ the inequality (45) is satisfied if $s_w < s_r < \frac{n+\delta}{A a^{\frac{1}{2\rho}}}$. Second, if $s_w > s_r$ the inequality is satisfied if $s_r < s_w < \frac{n+\delta}{A a^{\frac{1}{2\rho}}}$. □

If the elasticity of substitution between inputs is greater than one then independently of the savings rates no fluctuation occurs, whereas, for $\sigma < 1$ fluctuations can only occur if the savings rate of shareholders is larger than the savings rate of workers. For $s_w \geq s_r$ dynamics can only converge or diverge and the growth model can not generate complex dynamics.

4.4 Complex attractors

In this section we analyze the qualitative asymptotic properties of F in the case of elasticity of substitution between inputs lower than one, by using numerical simulations. Generic trajectories may converge to a fixed point or to a more complex attractor, that could be periodic or chaotic. Moreover, we will show that many coexisting attractors may emerge.

In order to assess the possibility of complex dynamics arising, we have to consider the case in which $\rho < 0$ and shareholders save more than workers. This fact proves that fluctuations and more complex dynamics in economic growth are influenced by the elasticity of substitution and the saving propensities of agents. The eigenvalue associated to an attractor Z of map F can be negative if $\sigma(k) < 1$ and $s_r > s_w$. Given the shape of map F we can not determine conditions for transition to chaos, however we can describe it with numerical simulations, following the work of Brianzoni *et al.* [16] for a bimodal map.

Notice that F can be bimodal so that it admits a trapping interval bounded by the local minimum of F and his image .

Proposition 4.13. *Let F be given in (37), $\rho < 0$ and, $s_r > s_w$. Assume F has a local maximum k_M and a local minimum k_m with $k_M < k_m$. Then if $F(k_M) > k_M$ and $F(k_m) < k_m$ the set $R = [F(k_m), F(F(k_m))]$ is trapping.*

Being R a close invariant region, the attractor Z must belong to it. Furthermore Z can be a fixed point k_Z or a more complex set. Observe that the fixed point k_Z may lose stability only via period-doubling bifurcation. Notice that if $F(k_M) > k_m$ and $F(k_m) > k_M$ then $k_M \notin R$ while $k_m \in R$, i.e.

only the local minimum point of F belongs to R and the attractor Z can be a fixed point or a more complex attractor (see Figure 17). The only route to chaos is via period-doubling bifurcation.

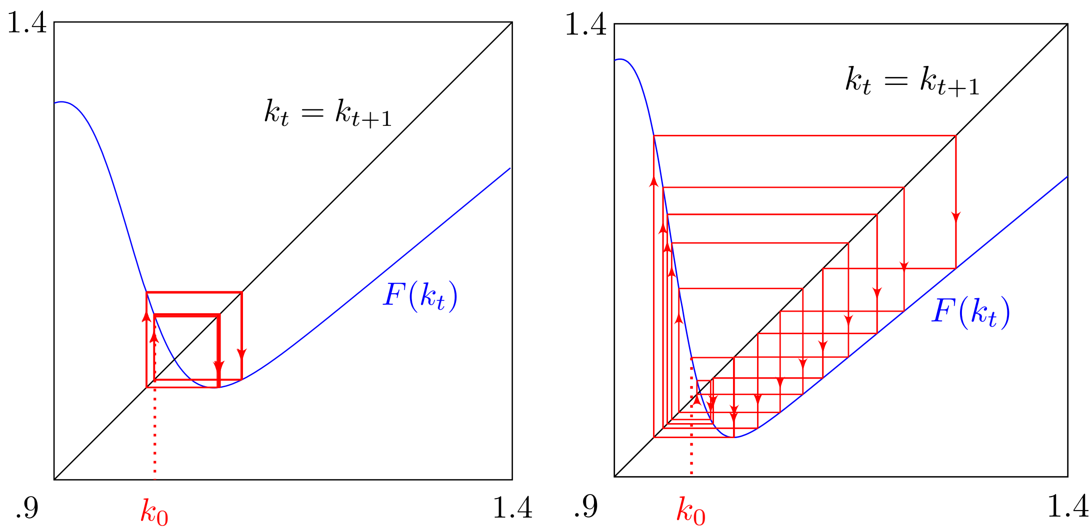


Figure 17: Parameter values $n = 0.2$, $\delta = 0.01$, $A = 0.9$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $s_w = 0.115$, $s_r = 0.86$. (a) 4-period cycle for $\rho = -35$. (b) Complexity emerges for $\rho = -60$.

As we have discussed, we are interested in the role played by the elasticity of substitution and the differential savings in order to obtain complex dynamics. Figure 18 shows that the dynamics are complex if the elasticity of substitution is low enough. As ρ decreases, the fixed point lose stability and a period doubling rout to chaos occurs. Therefore, the economic evolution may fluctuate if the elasticity of substitution between production factors is lower than one. We also consider the role of the difference between saving propensities as a determinant for cycle or chaos in the model, assuming $\Delta_s = s_r - s_w$ with $0 \leq \Delta_s \leq 1 - s_w$. The system becomes more complex as the difference between savings increases (Figure 19) and many period doubling and period halving cascades exist (see Hommes [34]).
In Figure 20 two cycle cartograms show different two-parametric bifurcation diagram qualitatively. In each cartogram, every color represents a long-run dynamic behaviour for a generic initial condition.

In panel (a) we consider the role of s_r and s_w. As we proved, complex dynamics may arise only if shareholders save more than workers. Observe that cycles of different order and complex dynamics (red region) arise for low levels of s_w, as s_r increases. In panel (b) we consider parameter ρ related to the elasticity of substitution and the difference between saving propensities Δ_s.

The bifurcation diagram show that dynamics are increasingly complex as the elasticity of substitution between production factors decreases and the difference between savings rates increases, confirming that fluctuations in economic growth models are influenced by saving propensities and production technology.

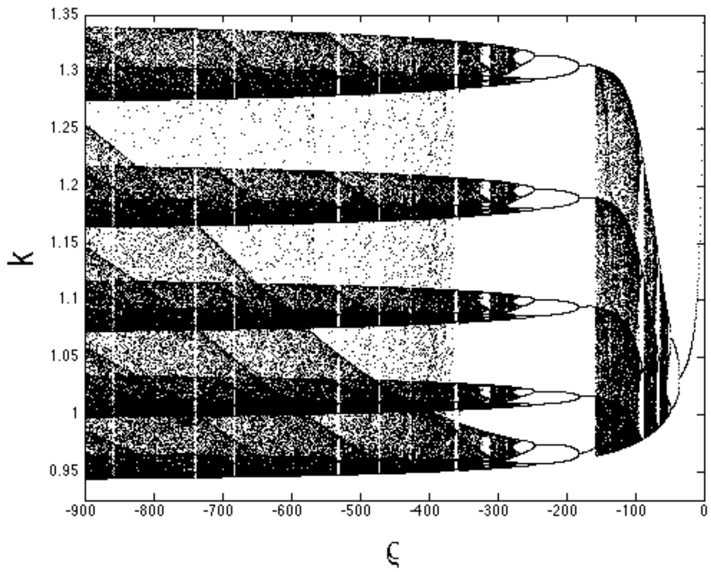


Figure 18: Bifurcation diagram w.r.t ρ. Parameter values $n = 0.2$, $\delta = 0.01$, $A = 0.9$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $s_w = 0.15$, $s_r = 0.7$.

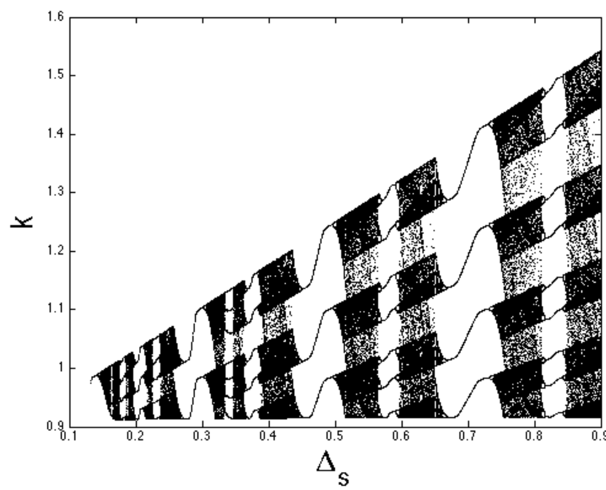


Figure 19: Bifurcation diagram w.r.t Δ_s. Parameter values $n = 0.2$, $\delta = 0.01$, $A = 0.9$, $a = 0.1$, $b = 0.25$, $c = 0.4$, $s_w = 0.1$, $\rho = -300$.

Proposition 4.7 implies that multiple equilibria may coexist and hence multistability phenomena may occur when the elasticity of substitution between capital and labour is lower than one (for multistability see Bischi *et al.* [10]). In Figure 21 we present two maps showing multistability

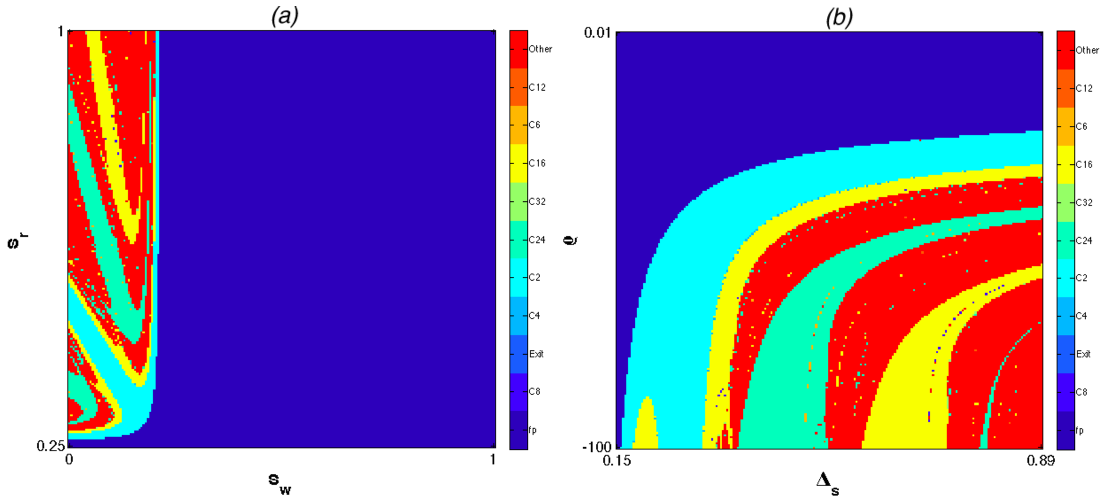


Figure 20: Parameter values: $n = 0.2$, $\delta = 0.01$, $A = 0.9$, $a = 0.1$, $b = 0.25$, $c = 0.4$. (a) Cycle cartogram in (s_w, s_r) plane for $\rho = -100$. (b) Cycle cartogram in (Δ_s, ρ) plane for $s_w = 0.11$.

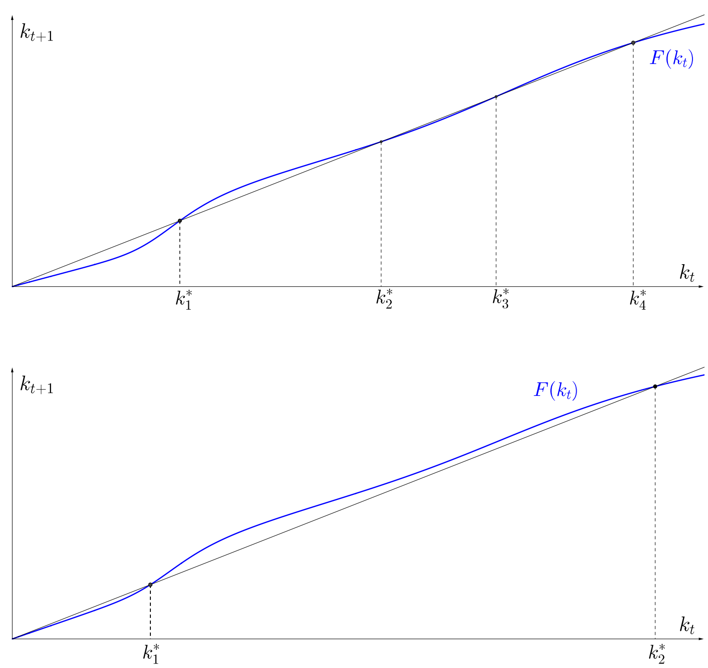


Figure 21: Multistability phenomenon for $n = .33$, $\delta = .56$, $A = 2.9$, $a = .02$, $b = .49$, $c = .01$, $s_w = .56$, $\rho = -7.1$, $s_r = .1$ (4 positive fixed points) $s_r = .19$ (two positive fixed points)

phenomena. Notice that in case of coexisting attractors, different initial conditions will converge to different equilibria. Therefore the neoclassical growth model with differential saving that take into consideration a production function with elasticity of substitution symmetric to input factors can explain the co-existence of non-developed, developing and developed economies. The existence of multiple steady states is due to the savings behaviour of workers and shareholders: as Section 3 and Section 4 discussed, low level of the savings rate of shareholders influence the early stage of the growth path and may generate poverty trap while higher equilibria are influenced by investments made from workers. When a multistability phenomena exists, an economic policy trying to increase investment may be able to push the economy to the higher capital per-capita equilibrium level.

4.5 Further developments

As a further step it should be analyzed how the growth path of the Kaldor model is bounded considering a general production function in order to provide conditions to mark boundaries of capital per capita levels during boom and bust cycles. Moreover, it should be investigated how savings rate of workers and shareholders influence growth bounds for non-developed, developing and developed countries. Conditions for boundedness of the accumulation law considering general and specific production functions should be given in order to analyze which savings behaviour influence the capital intensity of non-developed, developing and developed economies.

5 Conclusions

In this thesis the influence of elasticity of substitution between production factors on economic growth is investigated.
In the first chapter a method to measure the elasticity of substitution associated to an attractor is proposed in order to compare models with VES, sigmoidal and CES production functions and the relation between elasticity of substitution, capital per-capita and output levels. Moreover, the Solow model with differential savings assuming a VES production function, as proposed in Brianzoni et al. [19], is considered to verify whether the main result obtained by Miyagiwa and Papageorgiou [53] still holds also when using a variable elasticity of substitution production function. It is found that that, when the attractor is a fixed point, a higher elasticity of substitution is linked to a higher capital per-capital level in the steady state. This result fits with that obtained by Klump and La Grandville [42] using a CES production function. On the other hand, if more complex dynamics are exhibited, a negative correlation between elasticity of substitution and capital per-capita levels associated to the attractor may emerge, repeating the behaviour observed in Miyagiwa and Papageorgiou [53] for a sufficiently high elasticity of substitution. Evidence seems to indicate that, when the long run dynamics are simple, then there exists a positive correlation between elasticity of substitution, capital and output per-capita associated to the attractor. Therefore, when simple dynamics are exhibited, a higher elasticity of substitution causes a better efficiency of capital and labor: starting from a fixed level of capital per-capita, a country with a higher elasticity of substitution will experience a higher output per-capita in the equilibrium level and hence a greater economic growth. On the other hand, when the map exhibits cycles or more complex dynamics, then an ambiguous relation between elasticity of substitution and the asymptotic dynamics is shown. As a further development in this research line, the case in which the elasticity of substitution is not a linear function of k will be analyzed. In this case, the elasticity of substitution associated to the attractor cannot be calculated using using the simplification proposed in this work.
In the second chapter the Solow-Swan growth model with differential saving rate between workers and shareholders (see Böhm and Kaas [12], Kaldor [38, 37] and Pasinetti [56]) is studied using the Shifted Cobb-Douglas production function (see Capasso et al. [20]), a VES technology that well describe non developed, developing and developed economies. The results of the analysis shows that fluctuations or even chaotic patterns can be exhibited by the model confirming those obtained by Brianzoni et al. [14, 18, 16, 19, 17]: cycles and more complex dynamics may arise if shareholders save more than workers and the elasticity of substitution between production factors is lower than one. As in Brianzoni et al. [16] the system may converge to the poverty trap since the origin is always a locally stable fixed point, furthermore up to three positive fixed point may exists. Moreover, as in Klump and La Grandville [42], a positive correlation between elasticity of substitution al capital per-capita equilibrium level holds. A further development would be the analysis of multistability phenomena and related existence of complex basins. It is shown that if capital and labour are not easily replaceable, the growth path is bounded from above, moreover the boundary for low level of k is independent from the savings rate of workers; on the contrary, for high level of k, the boundary is independent from the saving behaviour of shareholders. Moreover, since multistability phenomenon may exist, the model described the coexistence of non-developed, developing and developed countries. Futhermore, an economy can lie in the poverty trap if investment of shareholders is not sufficiently high, regardless of the technology implied. As in Brianzoni *et al.* [19] it is found that when the elasticity of substitution is greater one only simple dynamics can be exhibited and unbounded endogenous growth is possible. Moreover, as in Karagiannis *et al.*[40], at most one

fixed point can exist. The results of the analysis show that fluctuations and even chaotic patterns may arise when the elasticity of substitution is lower than one and shareholders save more than workers, confirming that the elasticity of substitution between production factors plays a crucial role in economic growth theory (see, among all, Brianzoni *et al.* [18, 15, 14], Klump and de La Grandville [42] and Miyagiwa and Papageorgiou [53]).

References

[1] A. Agliari, L. Gardini, and T. Puu. "Global Bifurcations in Duopoly when the Cournot Point is Destabilized through a Subcritical Neimark Bifuraction". In: *International Game Theory Review* 8.1 (2006), pp. 1–20.

[2] R. G. D. Allen. *Mathematical Economics*. Macmillan, 1956.

[3] A. Andrikopoulos, J. Brox, and C. Paraskevopoulos. "Interfuel and Interfactor Substitution in Ontario Manufacturing, 1962-1982". In: *Applied Economics* 21 (1989), pp. 1–15.

[4] K.J. Arrow et al. "Capital-Labor Substitution and Economic Efficiency". In: *The Review of Economics and Statistics* 43.3 (1961), pp. 225–50.

[5] D. K. Arrowsmith and C. M. Palace. *Dynamical Systems: Differential Equations, Maps, and Chaotic Behaviour*. Chapman and Hall, 1992.

[6] C. Azariadis and J. Stachurski. "Poverty traps". In: *Handbook of Economic Growth*. Ed. by P. Aghion and Durlauf S.N. Vol. 1. Elsevier, 2005. Chap. 5, pp. 294–384.

[7] Erkin Bairam. "Capital-labor Substitution and Slowdown in Soviet Economic Growth: A Re-examination". In: *Bulletin of Economic Research* 42.63-72 (1990).

[8] Erkin Bairam. "Learning-by-doing, Variable Elasticity of Substitution and Economic Growth in Japan, 1878-1939". In: *Journal of Development Studies* 25.344-353 (1989).

[9] R.J.H. Beverton and S.J. Holt. "On the dynamics of exploited fish populations". In: *Fishery Invest.* 19 (1957), pp. 1–533.

[10] G.I. Bischi, C. Mammana, and L. Gardini. "Multistability and cyclic attractors in duopoly games". In: *Chaos, Solitons and Fractals* 11.4 (2000), pp. 543–564.

[11] V. Böhm. *Macroeconomic Theory*. Springer, forthcoming 2017.

[12] V. Böhm and L. Kaas. "Differential savings, factor shares, and en- dogenous growth cycles". In: *Journal of Economic Dynamics and Control* 24 (2000), pp. 965–980.

[13] V. Böhm and J. Wenzelburger. "Expectations, Forecasting, and Perfect Foresight – A dynamical Systems Approach". In: *Macroeconomic Dynamics* 3.2 (1999), pp. 167–186.

[14] S. Brianzoni, C. Mammana, and E. Michetti. "Complex Dynamics in the Neoclassical Growth Model with Differential Savings and Non-Constant Labor Force Growth". In: *Studies in Nonlinear Dynamics and Econometrics* 11.3 (2007).

[15] S. Brianzoni, C. Mammana, and E. Michetti. "Global attractor in Solow growth model with differential savings and endogenic labor force growth". In: *Modelling Measurement and Control D* 29.2 (2008), pp. 19–37.

[16] S. Brianzoni, C. Mammana, and E. Michetti. "Local and Global Dynamics in a Discrete Time Growth Model with Nonconcave Production Function". In: *Discrete Dynamics in Nature and Society* (2012).

[17] S. Brianzoni, C. Mammana, and E. Michetti. "Local and Global Dynamics in a Neoclassical Growth Model with Nonconcave Production Function and Nonconstant Population Growth Rate". In: *SIAM Journal of Applied Mathematics* 75.1 (2015), pp. 61–74.

[18] S. Brianzoni, C. Mammana, and E. Michetti. "Nonlinear dynamics in a business-cycle model with logistic population growth". In: *Chaos, Solitons and Fractals* 40 (2009), pp. 717–730.

[19] S. Brianzoni, C. Mammana, and E. Michetti. “Variable elasticity of substituition in a discrete time Solow–Swan growth model with differential saving”. In: *Chaos, Solitons and Fractals* 45 (2012), pp. 98–108.

[20] V. Capasso, R. Engbers, and D. La Torre. “On a spatial Solow model with technological diffusion and nonconcave production function”. In: *Nonlinear Anal.* 11 (2010), pp. 3858–3876.

[21] S. Chakraborty. “Endogenous lifetime and economic growth”. In: *Journal of Economic Theory* 116 (2004), p. 137.

[22] D. Cheban, C. Mammana, and E. Michetti. “Global attractors of non autonomous difference equations”. In: *Working paper* (2007).

[23] N. Coman. “Cournot competition in a monetary economy with overlapping generations of consumers: Equilibrium versus disequilibrim dynamics”. MA thesis. Department of Economics, Bielefeld University, 2011.

[24] W. D. Dechert. “Does optimal growth preclude chaos? A theorem on monotonicity”. In: *Zeitschrift für Nationalökonomie* 44 (1984), pp. 57–61.

[25] Robert L. Devaney. *An Introduction to Chaotich Dynamical Systems.* II edition. Westview Press, 2003.

[26] P. A. Diamond. “National Debt in a Neoclassical Growth Model”. In: *American Economic Review* 55.5 (1965), pp. 1126–1150.

[27] Romesh K. Diwan. “About the Growth Path of Firms”. In: *American Economic Review* 60.30-43 (1970).

[28] E. D. Domar. “Capital Expansion, Rate of Growth, and Employment”. In: *Econometrica* 14.2 (1946), pp. 137–147.

[29] F. Grassetti, C. Mammana, and E. Michetti. “Variable Elasticity of Substitution in the Diamond Model: Dynamics and Comparisons”. In: *Chaotic Modeling and Simulation Journal* 4 (2015), pp. 265–275.

[30] John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, 1983.

[31] K. Hamilton et al. *Where Is the Wealth of Nations?* The World Bank, 2005.

[32] R. F. Harrod. “An Essay in Dynamic Theory”. In: *The Economic Journal* 49.193 (1939), pp. 14–33.

[33] John Hicks. *The Theory of Wages.* Macmillan, 1932.

[34] C. Hommes. “Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand”. In: *Journal of Economic Behavior and Organization* 24.3 (1994), pp. 315–335.

[35] A. Jurgen. “Technical Change and the Elasticity of Factor Substitution”. In: *Beitrage der Hochschule Pforzheim* 147 (2014).

[36] K.R. Kadyiala. “Production functions and elasticity of substitution”. In: *The Southern Economic Journal* 38.3 (1972), pp. 281–284.

[37] N. Kaldor. “A model of economic growth”. In: *The Economic Journal* 67 (1957), pp. 591–624.

[38] N. Kaldor. “Alternative theories of distribution”. In: *The Review of Economic Studies* 23 (1956), pp. 83–100.

[39] N. Kaldor. "Marginal Productivity and the Macro-Economic Theories of Distribution: Comment on Marginal Productivity and the Macro-Economic Theories of Distribution: Comment on Samuelson and Modigliani". In: *The Review of Economic Studies* 33.4 (1966), pp. 309–319.

[40] G. Karagiannis, T. Palivos, and C. Papageorgiou. "Variable Elasticity Variable Elasticity of Substitution and Economic Growth". In: *Diebolt, Claude and Catherine Kyrtsou (eds): New Trends in Macroeconomics* (2005), pp. 21–37.

[41] Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[42] R. Klump and O. de La Grandville. "Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions". In: *American Economic Review* 90.1 (2000), pp. 282–291.

[43] R. Klump and H. Preissler. "CES Production Functions and Economic Growth". In: *The Scandinavian Journal of Economics* 102.1 (2000), pp. 41–56.

[44] C. A. Knox Lovell. "Capacity Utilization and Production function Estimation in Postwar American Manufacturing". In: *Quarterly Journal of Economics* 82.219-239 (1968).

[45] Y. Lu and L. Fletcher. "A Generalization of the CES Production Function". In: *The Review of Economics and Statistics* 50.4 (1968), pp. 449–452.

[46] R. Makrooni, F. Khellat, and L. Gardini. "Border collision and fold bifurcations in a family of one-dimensional discontinuous piecewise smooth maps: divergence and bounded dynamics". In: *Journal of Difference Equations and Applications* (2015).

[47] W. H. Masanjala and C. Papageorgiou. "The Solow Model with CES Technology: Nonlinearities and The Solow Model with CES Technology: Nonlinearities and Parameter Heterogeneity". In: *Journal of Applied Econometrics* 19.2 (2004), pp. 171–201.

[48] D. McFadden. "Constant Elasticity of Substitution Production Functions". In: *The Review of Economic Studies* 30.2 (1963), pp. 73–83.

[49] Alfredo Medio and Marji Lines. *Nonlinear dynamics. A primer.* Cambridge University Press, 2001.

[50] R. A. Meyer and K. R. Kadiyala. "Linear and Nonlinear Estimation of Production Functions". In: *Southern Economic Association* 40.463-472 (1974).

[51] E. Michetti. "Complex attractors and basins in a growth model with nonconcave production function and logistic population growth rate". In: *Mathematics and Computers in Simulation* (2013).

[52] Christian Mira et al. *Chaotic Dynamics in Two-Dimensional Noninvertible Maps*. Vol. 20. A. World Scientific, 1996.

[53] K. Miyagiwa and C. Papageorgiou. "Elasticity of substitution and growth: normalized CES in the Diamond model". In: *Economic Theory* 21.1 (2003), pp. 155–165.

[54] S. Nguyen and M. Streitwieser. "Capital-Energy Substitution Revisited: New Evidence from Micro Data". In: *Center for Economic Studies, U.S. Bureau of the Census, Working Paper* 4 (1997).

[55] C. Papageorgiou and M. Saam. "Two-level CES Production Technology in the Solow and Diamond Growth Models". In: *Scandinavian Journal of Economics* 110.1 (2008), pp. 119–143.

[56] L. L. Pasinetti. "Rate of profit and income distribution in relation to the rate of economic growth". In: *The Review of Economic Studies* 29 (1962), pp. 267–279.

[57] N. Paterson. "Elasticities of Substitution in Computable General Equilibrium Models". In: *Working paper, Department of Finance Canada* (2012).

[58] Lawrence Perko. *Differential Equations and Dynamical Systems*. Springer-Verlag, 1991.

[59] M. Prywas. "A Nested CES approach to Capital-Energy Substitution". In: *Energy Economics* 8.1 (1986), pp. 22–28.

[60] T. Puu. "Chaos in duopoly pricing". In: *Chaos, Solitons and Fractals* 1.6 (1991), pp. 573–581.

[61] T. Puu and I. Sushko, eds. *Oligopoly Dynamics. Models and Tools*. Springer-Verlag, 2002.

[62] N. S. Revankar. "A class of variable elasticity of substitution production function". In: *Econometrica* 39 (1971), pp. 61–71.

[63] N. S. Revankar. "Capital-Labor Substitution, Technological Change and Eco- nomic Growth: The U.S. Experience, 1929-1953". In: *Metroeconomica* 23 (1971), pp. 154–176.

[64] Clark Robinson. *Dynamical Systems. Stability, Symbolic Dynamics and Chaos*. CRC Press, 1999.

[65] Joan Robinson. "The Economics of Imperfect Competition". In: *Macmillan* (1933).

[66] R. Sato and R. F. Hoffman. "Production functions with variable elasticity of factor substitution: Some analysis and testing". In: *The Review of Economics and Statistics* 50.4 (1968), pp. 453–460.

[67] R. M. Solow. "A contribution to the theory of economic growth". In: *Quarterly Journal of Economics* 70 (1956), pp. 65–94.

[68] D. Stern. "Elasticities of Substitution and Complementarity". In: *Rensselaer Polytechnic Institute, Department of Economics, Rensselaer Working Papers in Economics* 0403 (2004).

[69] I. Sushko, A. Agliari, and L. Gardini. "Bistability and border-collision bifurcations for a family of unimodal piecewise smooth maps". In: *Discrete and Continuous Dynamical Systems B* 5.3 (2005), pp. 881–897.

[70] T. W. Swan. "Economic growth and capital accumulation." In: *Economic Record* 32 (1956), pp. 334–361.

[71] P. Thompson and T. Taylor. "The Capital-Energy Substitutability Debate: A New Look". In: *The Review of Economics and Statistics* 77.3 (1995), pp. 565–569.

[72] F. Tramontana, L. Gardini, and A. Agliari. "Endogenous cycles in discontinuous growth models". In: *Mathematics and Computers in Simulation* 81 (2011), pp. 1625–1639.

[73] F. Tramontana, F. Westerhoff, and L. Gardini. "A simple financial market model with chartists and fundamentalists: Market entry levels and discontinuities". In: *Mathematics and Computers in Simulation* 108 (2015), pp. 16–40.

[74] F. Tramontana, F. Westerhoff, and L. Gardini. "One-dimensional maps with two discontinuity points and three linear branches: mathematical lessons for understanding the dynamics of financial markets". In: *Decisions in Economics and Finance* 37 (2014), pp. 27–51.

[75] H. Uzawa. "Production Functions with Constant Elasticities of Substitution". In: *The Review of Economic Studies* 29.4 (1962), pp. 291–299.

[76] Stephen Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer-Verlag, 1990.