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ORIGINAL RESEARCH

Improving the quality of life and longevity of the elderly: the role of private versus public health

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Abstract
We develop an overlapping generations model to study how the combination of public and private health expenditures affects the health status and/or longevity of the elderly, as well as its impact on steady-state economic growth. We find that two distinct scenarios may arise-one with and one without reliance on private health care-depending on the relative value of private versus public health spending. In both cases, a positive locally asymptotically steady state emerges in terms of capital per worker, and a switch between regimes may occur depending on the share of public balance spent on the health system. Furthermore, increasing such a share increases the equilibrium longevity, while the effects on health status are ambiguous. Specifically, when the effectiveness of public expenditure is low, increasing public resources allocated to healthcare does not necessarily lead to improvements in health status. In contrast, when public spending is highly effective, greater allocation of public resources becomes a powerful tool to improve health status in old age.

Keywords Public and private health expenditures · Longevity · Life quality of the elderly · Economic growth · Overlapping generations model

1 Introduction

In recent decades, nearly all countries have experienced a substantial increase in human longevity. Life expectancy estimates the average number of years a person is expected to live based on various factors such as their birth year, current age, and demographic characteristics. It is a key indicator used to assess the general health and well-being of populations, often reflecting the effectiveness of healthcare systems, socioeconomic conditions, lifestyle choices,

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and public health policies. Between 1990 and 2022, life expectancy improved in all regions of the world, leading to a reduction in disparities between countries. In 2022, Chad (52.997 years), Lesotho (53.036 years), and Nigeria (53.633 years) had the lowest life expectancy at birth (LEB), while Japan (84.82 years), Liechtenstein (84.656 years) and Switzerland (84.255 years) reported the highest LEB. In 1990, the difference between the highest and lowest LEB-Japan and South Sudan, respectively-was 49.049 years (Aytemiz et al., 2024).

By 2022, this gap had narrowed to 31.823 years between Japan and Chad. Despite this progress, the remaining disparity is still substantial, highlighting the need to focus on achieving the Sustainable Development Goals (SDGs) set forth in the 2030 Agenda for Sustainable Development. (United Nations, 2024)[1]

Longevity has been a cornerstone in public health and medical debates and life expectancy is often used to measure the health and well-being of a population (Annas & Galea, 2018). At least in the developed world, longer expected lifespans have been accompanied by a significant increase in healthcare expenditures. A growing body of literature shows a clear link between health expenditures and health outcomes (Glied & Smith, 2013). In particular, data from advanced countries show that public health spending plays a very important role in enhancing longevity, a much greater role than private health spending. This evidence stems from the fact that: *"public health expenditure is devoted, first and foremost, to finance actions that affect an important fraction of the population and involve significant positive external effects (i.e., what we could identify as basic health: vaccination campaigns, prevention of disease, basic framework of health centers, etc.). But once basic programs are met, additional public expenditure is likely to be devoted to activities that the private sector also offers, so the productivities of the two sectors converge."* Aísa et al. (2014).

In fact, there are some countries, e.g. the United States, where healthcare expenditure is high, but the public component of this expenditure is low, with limited effects on improving longevity. With regard to this, in 2023, the life expectancy in the United States was 78.4 years, which is 4.1 years lower than the average of comparable countries at 82.5 years. As Aísa et al. (2014) states: "*In particular, the above results offer a plausible explanation for the apparently paradoxical data for the USA. While this is the country that devotes the largest amount of resources to health (over 13 % of GDP, close to twice the average value in the sample), life expectancy was 76.2 years in 2000, below the average. The key element that enables us to understand this puzzle is precisely the composition of health expenditure. The average value in the sample shows a ratio of public to private expenditures of 3.72. In contrast, this ratio is only 0.82 for the USA. That is, the public health system in the USA is responsible for only 45 % of the resources devoted to health, compared to the almost 80 % average in the OECD countries. The above results show the need for a redesign of the health system through an intensive promotion of the public health system.*"

Although life expectancy has generally increased, healthy life expectancy (HALE) has not progressed at the same rate.[2] Between 2000 and 2019, HALE increased by 5.3 years, from 58.1 to 63.5 years, while overall life expectancy grew by 6.4 years during the same period. This indicates that, although people are living longer, they are also spending more

[1] Available at: https://sdgs.un.org/goals..

[2] Healthy life expectancy is a measure that combines both quantity of life and health status. This refers to the average number of years that a person can expect to live in full health, free from disease or disability. It is not just about how long people live, but how long they live without major health problems.

of those additional years dealing with health problems.[3] But, as Zarulli and Caswell (2024) stress, surviving in good health is essential because healthy years of life are also a crucial part of the dynamics of the life cycle.

These trends underscore the importance of addressing not only longevity, but also health status during extended lifespans. Efforts to improve access to healthcare, preventive measures, and health education are crucial in ensuring that increased life expectancy translates into healthier years lived. Governments around the world recognize the importance of the health system; therefore, health expenditures throughout the world have increased with time.[4] Health expenditures are mostly financed through public taxation and are growing more than the global economy, accounting for 10% of the world gross domestic product (GDP) (25). The average health expenditures as a share of GDP have increased from 7.8% in 2005 to 9.8% in 2020 in the OECD countries (Anwar et al., 2023).

In our paper, we propose an economic framework able to account for both elements relevant to health: one related to life expectancy (i.e., life quantity), and one expressing quality from the point of view of the elderly person's health (i.e., health status). We therefore construct a 2-period overlapping-generations model that incorporates an analysis of endogenous longevity together with the endogenous elderly health status. Therefore, we combine two strands of research by considering both the 'quantity' of life and health status.

Following (Chakraborty, 2004) and Cipriani and Fioroni (2019), we assume that the probability of survival from the first period (adulthood) to the next (old age) is endogenously determined through public investment in health.[5]

It is also important that elderly survivors increase their health status. Hence, we assume, following (Varvarigos & Zakaria, 2013), that this state of health depends on both public expenditures on medical care, and also on private spending by individuals during the final period of their lives. In fact, the elderly can decide to allocate part of their income to health care expenditures to improve their utility-enhancing health status. Understanding how public and private health expenditures can interact is essential for shaping or reforming health care policies.

Bhattacharya and Qiao (2007) and Varvarigos and Zakaria (2013) consider that private and public health expenditures are complementary, i.e., an increase in one leads to an increase in the other, rather than substituting for it. This occurs when private and public investments reinforce each other in improving elderly health.Unlike(Varvarigos & Zakaria, 2013), we interpret the public component as a "competitor" to the private input in determining elderly health because public and private expenditures have a certain degree of substitutability.

The choice to consider both private and public expenditure by the elderly arises from the fact that private expenditure in this context - such as medical treatments and interventions

[3] GHE: Life expectancy and healthy life expectancy. The Global Health Observatory. https://www.who.int/data/gho/data/themes/mortality-and-global-health-estimates/ghe-life-expectancy-and-healthy-life-expectancy - World Health Organization (2022).

[4] WHO. Countries are Spending More on Health, But People are Still Paying Too Much Out of Their Own Pockets. WHO (2019). Available online at: https://www.who.int/news/item/20-02-2019-countries-are-spending-more-on-health-but-people-are-still-paying-too-much-out-of-their-own-pockets.

[5] The decision not to consider private spending as a determinant of longevity is based on the fact that data on developed countries, such as the United States, as highlighted in the introduction, show that public healthcare spending is the main determinant of a country's longevity. Furthermore, in our model, longevity is the probability of surviving into the second period, and therefore private spending on longevity should be done by young adults, which is not very common. Thus, this modelling makes the analytical model more manageable without implying a loss of generality of the model itself.

for existing diseases and conditions - tends to improve the health status of the elderly. These expenditures are primarily focused on care, as prevention is often not an immediate priority for private individuals who allocate their resources mainly toward treatments and therapies. As a result, this type of healthcare can be provided interchangeably by the public health system or, in cases of deficiencies or excessively long waiting times, by the private system.

In this regard, (Li et al., 2016) conduct a quantitative exercise to examine whether the observed differences in the public-private mix of health expenditure can be accounted for by variations in the elasticity of substitution between private and public spending, as well as by differences in the effectiveness of public expenditure in health production across a sample of OECD countries. Following the quantitative analysis by Li et al. (2016), our model considers two different scenarios regarding the effectiveness of public health expenditure, each representing two markedly different models of private-public health expenditure mixes.

This paper presents a simple framework to study the dynamic substitutability between public health programs and private efforts, aimed at improving health status in old age, given that longevity depends on the public health system.

By combining analytical methods and numerical experiments, our model shows that capital per young worker evolves over time according to different dynamics. Specifically, two scenarios may emerge: in the first the value of private health spending is higher than public spending, leading the elderly to allocate part of their savings to health; in the second the value of private health spending is lower than public spending, and the elderly choose not to invest in health. In both cases, the model admits a unique, positive, and locally asymptotically stable steady state. However, as the share of public resources allocated to health care changes, a switch between the two scenarios may occur. While this transition does not affect the equilibrium level of longevity, its impact on the health status may be ambiguous.

The paper proceeds as follows. Section 2 illustrates the set up of the model. Section 3 presents the dynamics of the system. Section 4 concludes.

2 The economic setup

2.1 Health status and quantity in the elderly

We consider a production economy populated by overlapping generations of agents who live for two periods: adulthood and old age. Following (Chakraborty, 2004), longevity is endogenous, i.e. the probability of surviving from adulthood to old age depends on public expenditures on health. Hence, we assume that longevity depends only on public health expenditures made by the government at time $t \in \mathbb{N}$, and the more the government invests in public healthcare, the greater the probability of an adult surviving as an elderly person.

We define as w_t the wage rate of an adult for a unit of effective labor, then, being $\tau \in (0, 1)$ the taxation rate, the public health financed by the state positively depends on fiscal revenues i.e., τw_t.

We denote by g_t the amount of public expenditure on health at time *t*, then, let $\gamma \in (0, 1)$ be the fraction of the public balance devoted to the health system, we obtain

$$g_t = \gamma \tau w_t. \tag{1}$$

Although the length of each period is normalized to one, an individual's lifetime is uncertain. More precisely, the individual probability of surviving from the first period (adulthood) to the next (old age) depends on public expenditures on health when young. We refer to this component as quantity of life in the elderly, and it is described by a variable measuring the survival probability of an adult, thus becoming old in the second period. We denote this probability as $p_t \in [\bar{p},\, 1)$, given a fixed exogenous threshold for longevity, which is associated with no investment in public health, denoted by $\bar{p} \in [0, 1)$.

Then, following (Chakraborty, 2004) and Cipriani and Fioroni (2019), we consider a function $P : \mathbb{R}_+ \to [\bar{p},\, 1)$ which associates the longevity p_t with public health expenditure g_t, such that, $P(0) = \bar{p}$. Specifically, there exists an exogenous level of longevity where $\lim_{g_t \to +\infty} P(g_t) \to 1$, meaning that if public health expenditure is sufficiently high, then longevity is close to one. The function P is continuous, twice differentiable and such that $P' > 0$, indicating that increased public expenditure on health improves longevity, and $P'' < 0$, reflecting diminishing marginal benefits.

A function satisfying these assumptions is as follows:

$$p_t = \bar{p} + \frac{(1-\bar{p})g_t}{1+g_t}. \tag{2}$$

Regarding the health status in the elderly, following (Varvarigos & Zakaria, 2013), we assume it is related to the expenditures in the health system through two components: the public one, financed by the state via taxation, and the private one, funded by the elderly themselves, who divert a portion of their resources from consumption in old age. However, unlike (Varvarigos & Zakaria, 2013), we consider private and public health expenditures to act as a perfect substitute to improve the health status of the elderly.

The substitutability between private and public health expenditures can be explained by the estimates of Li et al. (2016), which suggest that for most OECD countries, either the Cobb-Douglas form or a linear form is a reasonable representation for health technology. In fact, in old age individuals primarily allocates resources toward treatments and therapies aimed at managing the health effects of existing conditions. In this context, private spending may be a solution to the excessive waiting times in public healthcare and congestion problems related to public spending.

We denote private health expenditure in old age as x_{t+1}, which represents a fraction of the individual's savings. Thus, these savings are used both for consumption in old age and for improving health status.

Regarding the health status of the elderly, it is assumed that the health technology that determines the health status depends on both private and public health expenditures at time $t+1$, with a perfect degree of substitutability between them.[6]

The health status in old age, denoted by h_{t+1}, can be formalized as follows:

$$h_{t+1} = \alpha g_{t+1} + (1-\alpha)x_{t+1}. \tag{3}$$

[6] Considering imperfect substitutability between public and private spending on health technology (considering, for example, a Cobb Douglas function) implies that the technical substitution rate between the two types of spending is not constant but decreases as the use of one of the two types of spending increases. As we already mentioned, based on the empirical analysis of Li et al. (2016), both types of function (linear or Cobb Douglas) are supported by empirical data, but the use of a Cobb Douglas function would make the theoretical analysis much more complex.

where $\alpha \in (0, 1)$ represents the effectiveness of public expenditure in improving the health status of the elderly, while $(1 - \alpha)$ is the effectiveness of private expenditure by the elderly themselves. The ratio $\frac{\alpha}{(1-\alpha)}$ measures the rate at which private and public health expenditure can be exchanged, while maintaining a constant level of health.

2.2 Intertemporal constrained utility maximization

We consider an economy consisting of an infinite sequence of overlapping generations, each potentially living for two periods, alongside an infinitely-lived government. Time is discrete, that is, $t = 1, 2, ...$ Following (Chakraborty, 2004), we assume that each individual born in generation *t* gives birth to one offspring at the end of period *t*, before experiencing their mortality shock. The new individual becomes economically active only at the beginning of $t + 1$. Therefore, in each period a measure-one cohort of adults is born, each endowed with one unit of time, which they inelastically supply to the labor market, earning a wage income w_t. Following (Cipriani & Fioroni, 2019), to keep the model more tractable, we assume that all adults retire at the end of the first period.

In summary, adults benefit from the consumptions during adulthood, consumptions in old age, and their health status h_{t+1} when old, depending on the survival probability. The lifetime utility of an individual of generation *t* is then given by the following function[7]:

$$U_t = \ln c_t + p_t\{\beta \ln c_{t+1} + \theta \ln h_{t+1}\}, \tag{4}$$

where c_t is consumption during adulthood, c_{t+1} is consumption during old age, and h_{t+1} is the health status during old age. The parameters β, θ are positive, and p_t represents the probability of surviving from youth to old age.

The budget constraint in the first young adulthood period is given by:

$$(1 - \tau)w_t = c_t + s_t \tag{5}$$

where $\tau \in (0, 1)$ is the tax rate on labor income and s_t refers to savings (understood as the purchase of annuities). All variables are assumed to be non-negative.

Given the adults' salary and the interest rate, the cost of health expenditure reduces the resources available for both future consumption and savings.

The second-period budget constraint is given by:

$$s_t\hat{R}_{t+1} = c_{t+1} + x_{t+1} \tag{6}$$

where x_{t+1} is the private expenditure for health status, and $\hat{R}_{t+1}$ is the gross return on its savings. In fact, consumption in old age is financed by the returns on savings accumulated during adulthood. Furthermore, we assume that all goods are perishable and that agents can

[7] The assumption of additivity in the utility function in overlapping generations models is often used (see, for example (De La Croix & Michel, 2002)). Additivity implies that the well-being derived from consumption does not depend on the level of health and vice versa. In reality, consumption is often more "useful" if you are healthy and vice versa: in fact, greater consumption can improve health status. However, this hypothesis allows us to analyze choices relating to consumption and health separately: in fact, the agent decides to invest part of their savings in health only by looking at the direct contribution to utility, without considering how health affects the marginal utility of consumption.

only transfer value over time through capital markets. Individuals are assumed to have no bequest motives. Following (Chakraborty, 2004), in order to eliminate the risks associated with uncertain lifespans, we assume the existence of a perfect annuity market, where all savings are managed through mutual funds. At the end of their youth, individuals deposit their savings into a mutual fund. These funds are exclusively invested in capital, and the mutual fund guarantees a gross return to those who survive into old age. If these funds yield a gross return of $\hat{R}_{t+1}$ on its investment, then under perfect competition, equilibrium in the annuity market is maintained:

$$\hat{R}_{t+1} = \frac{R_{t+1}}{p_t}, \tag{7}$$

where R_{t+1} is the gross interest rate.

To summarize, we can collect all the equations presented so far to formulate the following constrained optimization problem:

$$\max \quad U_t = \ln c_t + \beta p_t \ln c_{t+1} + \theta p_t \ln h_{t+1} \tag{8}$$

$$\text{s.t.:} \quad g_{t+1} = \gamma \tau w_{t+1} \tag{9}$$

$$c_t + s_t = (1-\tau) w_t \tag{10}$$

$$c_{t+1} + x_{t+1} = s_t \hat{R}_{t+1} \tag{11}$$

$$\hat{R}_{t+1} = \frac{R_{t+1}}{p_t} \tag{12}$$

$$h_{t+1} = \alpha g_{t+1} + (1-\alpha) x_{t+1} \tag{13}$$

$$x_{t+1} \geq 0. \tag{14}$$

The solution of the maximum constrained optimization problem results in the following Proposition.

Proposition 2.1 *The first-order conditions for problem (8–14) are as follows.*

Case A: *If*

$$(1-\alpha)\theta R_{t+1}(1-\tau) w_t \geq (1+\beta p_t)\alpha\gamma\tau w_{t+1} \tag{15}$$

then

$$c_t = \frac{1}{1+(\beta+\theta)p_t}\left[\frac{p_t}{R_{t+1}}\frac{\alpha}{1-\alpha}\gamma\tau w_{t+1} + (1-\tau)w_t\right], \tag{16}$$

$$c_{t+1} = \frac{\beta R_{t+1}}{1+(\beta+\theta)p_t}\left[\frac{p_t}{R_{t+1}}\frac{\alpha}{1-\alpha}\gamma\tau w_{t+1} + (1-\tau)w_t\right], \tag{17}$$

$$x_{t+1} = \frac{1}{1+(\beta+\theta)p_t}\left[\theta R_{t+1}(1-\tau)w_t - (1+\beta p_t)\frac{\alpha}{1-\alpha}\gamma\tau w_{t+1}\right], \tag{18}$$

$$s_t = \frac{p_t}{1+(\beta+\theta)p_t}\left[(\beta+\theta)(1-\tau)w_t - \frac{1}{R_{t+1}}\frac{\alpha}{1-\alpha}\gamma\tau w_{t+1}\right]. \tag{19}$$

Case B: *If*

$$(1-\alpha)\theta R_{t+1}(1-\tau)w_t < (1+\beta p_t)\alpha\gamma\tau w_{t+1} \tag{20}$$

then

$$c_t = \frac{(1-\tau)w_t}{1+\beta p_t}, \tag{21}$$

$$c_{t+1} = \frac{\beta R_{t+1}(1-\tau)w_t}{1+\beta p_t}, \tag{22}$$

$$x_{t+1} = 0, \tag{23}$$

$$s_t = \frac{\beta p_t(1-\tau)w_t}{1+\beta p_t}. \tag{24}$$

Moreover, in both cases, these conditions are also sufficient.

Proof See Appendix A. □

The previous conditions referred to Case A and Case B become easier to interpret economically. In fact, the condition relating to Case A is:

$$(1-\alpha)\theta R_{t+1}(1-\tau)w_t \geq (1+\beta p_t)\alpha g_{t+1}. \tag{25}$$

The left-hand side of the inequality represents the value (θ) of the (net, discounted) income spent on private expenditure when elderly (also taking into account the effectiveness $1-\alpha$), while the right-hand side, on the other hand, represents the value of public expenditure g_{t+1} of which the effectiveness α is also taken into account. Therefore, when the inequality of Case A is verified, it means that the value of private health expenditure is greater than public expenditure. As a consequence, private spending on health by the elderly is positive.

Conversely, in Case B the value of private spending on health is lower than public spending, and therefore the elderly will not invest their savings in health spending, i. e. $x_{t+1} = 0$.

Thanks to the first-order conditions, we can also derive h_{t+1} by substituting the expression of x_{t+1} in both cases A and B (respectively given by (18) and by (23)) into the formula (3) and considering that $g_{t+1} = \gamma\tau w_{t+1}$. After some algebraic manipulations, the expression for h_{t+1} in Case A is as follows:

$$h_{t+1}^A = \frac{[\alpha\gamma\tau w_{t+1}p_t + (1-\alpha)(1-\tau)R_{t+1}w_t]\,\theta}{1+(\beta+\theta)p_t}, \tag{26}$$

while for Case B, we obtain:

$$h_{t+1}^B = \alpha\gamma\tau w_{t+1}. \tag{27}$$

It is straightforward to observe that $h_{t+1}^A \geq h_{t+1}^B$. In fact, from (13), (9), and (14) we have:

$$h_{t+1} = \alpha g_{t+1} + (1-\alpha)x_{t+1} \geq \alpha g_{t+1} = \alpha\gamma\tau w_{t+1} = h_{t+1}^B. \tag{28}$$

In particular, (28) holds if h_{t+1}^A is substituted on the left-hand side. Alternatively, one arrives at a similar conclusion by bounding (26) using (15). In fact, from (15) we have:

$$\begin{aligned} h_{t+1}^A =& \frac{\alpha\gamma\tau w_{t+1}p_t\theta + (1-\alpha)(1-\tau)R_{t+1}w_t\theta}{1+(\beta+\theta)p_t} \\ \geq& \frac{\alpha\gamma\tau w_{t+1}p_t\theta + (1+\beta p_t)\,\alpha\gamma\tau w_{t+1}}{1+(\beta+\theta)p_t} = \alpha\gamma\tau w_{t+1} = h_{t+1}^B. \end{aligned}$$

As for the expression of p_t, in both cases we find that by substituting (1) into (2), we obtain:

$$p_t = \bar{p} + \frac{(1-\bar{p})g_t}{1+g_t} = \bar{p} + \frac{(1-\bar{p})\gamma\tau w_t}{1+\gamma\tau w_t}. \tag{29}$$

Thus, it can be seen that the quantity of life, i.e. longevity, is equal in the two scenarios as it depends only on public spending; conversely, health status is higher in scenario A where the elderly invest part of their savings in improving their health.

2.3 Production, investment and saving

As previously mentioned, at each period, a new generation of adults, each with measure one, enters the economy. Each agent is endowed with one unit of labor during their youth and is compulsorily retired in old age.

The aggregate production technology of the economy is assumed to follow a constant-returns-to-scale production function, utilizing both labor and capital. Capital stock is assumed to fully depreciate after one period of use, meaning that the capital stock in any given period is equal to the savings in the previous period.

Production in time *t* employs physical capital K_t and labor L. We denote Y_t the aggregate output and represent the aggregate technology of the economy by the following production function:

$$Y_t = F(K_t, L) = AK_t^{\delta}L^{1-\delta}. \tag{30}$$

In that equation, Y_t, K_t, and L respectively stand for aggregate output, physical capital stock and effective labor in the economy in period *t*, while A is the total factor productivity, and $\delta \in (0,1)$ is the productivity of physical capital.

Defining the capital stock per young worker as $k_t = \frac{K_t}{L}$, the production function can be rewritten as

$$\frac{Y_t}{L}(\equiv y_t) \equiv Ak_t^{\delta}.$$

Perfect competition in the final goods market implies that the wage rate w_t and the interest rate r_t are equal to the marginal productivity of labor and capital, so that:

$$w_t = A(1-\delta)k_t^{\delta} \tag{31}$$

and

$$r_t = A\delta k_t^{\delta-1}, \tag{32}$$

and we obtain the following gross interest rate:

$$R_{t+1} = 1 + r_{t+1}. \tag{33}$$

From the capital market, the following equilibrium condition is assumed to hold:

$$k_{t+1} = s_t. \tag{34}$$

3 Qualitative and quantitative dynamics

To describe the evolution of the economy over time, we consider both the necessary and sufficient conditions of the constrained optimization problem as stated in Proposition 2.1, as well as the relationships among capital per young worker, the wage rate, and the gross interest rate under the assumption of capital market equilibrium, as presented in Sect. 2.3.

The capital per young worker over time is obtained in the following Proposition 3.1.

Proposition 3.1 *Let* $(k_t,\, k_{t+1}) \in \mathbb{R}^2_+$ *and*

$$p_t\left(k_t\right) = \bar{p} + \frac{\left(1-\bar{p}\right)\gamma\tau A(1-\delta)k_t^{\delta}}{1+\gamma\tau A(1-\delta)k_t^{\delta}}. \tag{35}$$

Define

$$f_C(k_t,\, k_{t+1}) = \left[\theta\left(1 + A\delta k_{t+1}^{\delta-1}\right)(1-\tau)k_t^{\delta} - (1+\beta p_t(k_t))\,\frac{\alpha}{1-\alpha}\gamma\tau k_{t+1}^{\delta}\right]A(1-\delta). \tag{36}$$

(a) *If* $f_C\left(k_t,\, k_{t+1}\right) \geq 0$*, then the following relationship between* k_t *and* k_{t+1} *must hold:*

$$f_A\left(k_t,\, k_{t+1}\right) := k_{t+1} - \frac{A(1-\delta)p_t\left(k_t\right)}{1+(\beta+\theta)p_t\left(k_t\right)}\left[(\beta+\theta)(1-\tau)k_t^{\delta} - \frac{1}{1+A\delta k_{t+1}^{\delta-1}}\frac{\alpha}{1-\alpha}\gamma\tau k_{t+1}^{\delta}\right] = 0. \tag{37}$$

(b) *Otherwise, if* $f_C(k_t, k_{t+1}) < 0$, *then the following relationship between* k_t *and* k_{t+1} *must hold:*

$$f_B(k_t, k_{t+1}) := k_{t+1} - \frac{\beta p_t(k_t)(1-\tau)A(1-\delta)k_t^{\delta}}{1+\beta p_t(k_t)} = 0. \tag{38}$$

Proof See Appendix B. □

Notice that condition $f_C(k_t, k_{t+1}) \geq 0$ refers to Case A of Proposition 2.1, where the attributed value of private health expenditure exceeds the value of public spending, resulting in positive private health investment by the elderly. Conversely, condition $f_C(k_t, k_{t+1}) < 0$ is related to Case B of Proposition 2.1, i.e. the value of private health spending is lower than the value of public spending, and therefore the elderly will not invest their savings in health. Proposition 3.1 outlines the conditions that determine which case point (k_t, k_{t+1}) belongs to and implicitly describes the evolution of capital per young worker over time in both cases, i.e. with or without private health expenditure.

The boundary equation $f_C(k_t, k_{t+1}) = 0$ as given by equation (36) separates the plane $\mathbb{R}_+^2$ into two regions, R_A and R_B, given by

$$R_A = \{(k_t, k_{t+1}) \in \mathbb{R}_+^2 : f_C(k_t, k_{t+1}) \geq 0\}, \tag{39}$$

where the evolution of capital per worker over time is fully described by the implicit equation $f_A(k_t, k_{t+1}) = 0$ as defined by equation (37), and

$$R_B = \{(k_t, k_{t+1}) \in \mathbb{R}_+^2 : f_C(k_t, k_{t+1}) < 0\}, \tag{40}$$

where the evolution of capital per worker over time is fully described by the implicit equation $f_B(k_t, k_{t+1}) = 0$ as defined by equation (38).

Hence, the shape of the curve $f_C(k_t, k_{t+1}) = 0$ on the plane plays an important role in the description of the dynamics shown by the proposed model.

Several numerical experiments indicate that the boundary equation passes through the origin and is strictly increasing and concave. Notice also that $f_C(0, k) < 0, \forall k > 0$, hence the $y-$axes belongs to R_B. As a consequence, the region R_A (resp. region R_B) corresponds to the set of points below the curve f_C (resp. above the curve f_C). This behavior is clarified in Fig. 1a, where the two regions are depicted in different colors (from now on we will associate the color cyan with Case A and magenta with Case B).

Analyzing the dynamics generated by the discrete-time equations defined by f_A in R_A and f_B in R_B is quite complex. This complexity arises both from the analytical form of the functions involved and from the fact that one of them is defined in an implicit form. As a result, a more thorough investigation of the existence and stability of steady-state equilibria requires a combination of analytical techniques and numerical methods.

With regard to the numerical experiments that will be conducted to support the analytical results, in Table 1 we present the values assigned to some of the parameters used in the model, while others may vary within their respective domains to facilitate a comparative

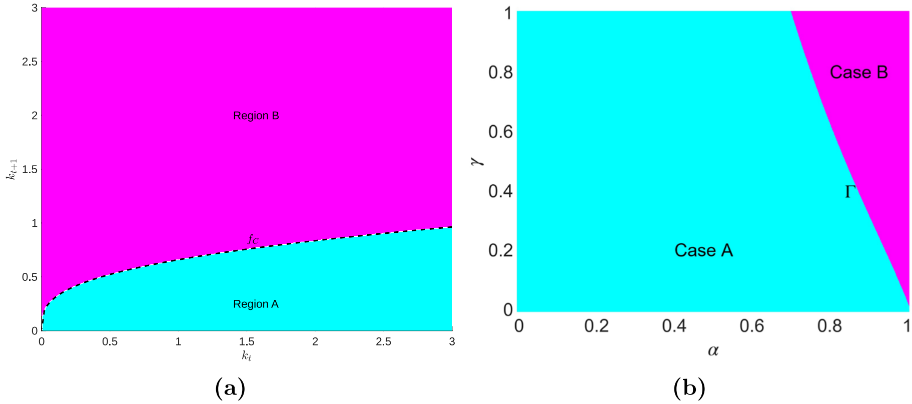


Fig. 1 **a** The border f_C that separates region A (in cyan) from region B (in magenta) is represented by a dashed black line. **b** The curve $\gamma = \Gamma(\alpha)$ that separates the combinations of parameters (α, γ) leading to Case A (in cyan) or Case B (in magenta) is obtained

Table 1 Values assigned to parameters in numerical experiments

Parameter values	References
$\bar{p} = 0.5$	Exogenous longevity level
$\tau = 0.3$	Average level of taxation of OECD countries
$\beta = \theta = 0.3$	Varvarigos and Zakaria (2013)
$\delta = 0.3$	Bhattacharya and Qiao (2007)
$A = 10$	Scale parameter

analysis. Several numerical experiments show that the qualitative results do not change when these parameters vary, and therefore the results described remain valid even for systems characterized by different values of A or $\bar{p}$, thus demonstrating their general validity and robustness.

Firstly, to focus on the role that the public health system can have on the quantity of life and health status in old age, we allow the parameter γ, which measures the share of the public budget allocated to the health system, to vary. This serves as a bifurcation parameter in our simulations. Secondly, we also consider the role of the parameter α, which measures the effectiveness of public spending in the health production function.

As previously mentioned, and in line with Li et al. (2016), we consider two distinct values of α, which correspond to two different models for the mix of private and public health expenditures. In particular, a value of $\alpha = 0, 5$, represents countries, such as the US, Switzerland, and France in which the effectiveness of public and private expenditure in the health production function is equal. In contrast, some countries, such as Denmark, Italy, New Zealand, Norway, and Sweden demonstrate greater effectiveness in public spending compared to private spending (a value of α approximately equal to 0, 8).

In Fig. 1b, we numerically obtain a curve given by $\gamma = \Gamma(\alpha)$ in the set $[0, 1] \times [0, 1]$, separating the parameter plane (α, γ) into two regions, one referred to the combinations leading to Case A of Proposition 2.1 (in cyan) and the other one referred to Case B of Proposition 2.1 (in magenta). Notice that, when we fix all parameters as shown in Table 1 and allow α and

γ to vary, if $\alpha \to 0^+$ or $\gamma \to 0^+$, the curve $f_C\,(k_t,\,k_{t+1}) = 0$ tends to the $y-$axis, meaning that the economic model is fully described by f_A.

The magenta case refers to a situation in which, once the values of the other parameters have been set, low values of γ and α -indicating insufficient allocation of tax revenue to public healthcare and low efficiency of public spending in the healthcare production function -lead individuals to allocate part of their savings to private healthcare expenditures, thereby shifting the system into Scenario A.

Conversely, if γ or α are not too low, the elderly place a higher value on public healthcare than on private healthcare. This results in Case B from Proposition 2.1, where no resources are allocated to the private healthcare system.

In the following analysis, we will study the qualitative and quantitative dynamics of the model defined in Proposition 3.1, distinguishing between the Case A (where the value of private healthcare expenditure exceeds the value of public expenditure and the elderly invest in private healthcare), and the Case B (where the value of private healthcare is lower than that of public spending, leading the elderly to not invest their savings in private healthcare), as outlined in Proposition 2.1.

3.1 Scenario A: private healthcare spending.

Consider the case where Proposition 3.1 point (*a*) is verified. In this case, the relationship between k_t and k_{t+1} must satisfy the implicit function f_A, as given by equation (37), within the domain R_A, as defined in equation (39). This refers to combinations of α and γ that belong to the region shaded in cyan in Fig. 1b.

While the shape of the curve $f_A(k_t,\,k_{t+1}) = 0$ in $\mathbb{R}^2_+$ cannot be derived analytically, numerous numerical experiments have been conducted, all of which confirm that f_A defines a function that passes through the origin, is strictly increasing, and concave.

Based on the numerical evidence summarized above, the following remark trivially holds.

Remark 3.2 The implicit function $f_A(k_t, k_{t+1})$ admits a unique positive fixed point $E_A(k^*_A, k^*_A)$ which is asymptotically stable. If $E_A \in R_A$, then it represents a feasible fixed point; otherwise, it is a virtual fixed point. Additionally, the point $E_0 = (0, 0)$ is an unstable fixed point.

The evidence presented in Remark 3.2 can be better understood by examining Fig. 2, which shows the evolution of capital per young worker over time, starting from a given initial condition. In these diagrams, the curves $f_A\,(k_t,\,k_{t+1}) = 0$ and $f_B\,(k_t,\,k_{t+1}) = 0$ are drawn with solid lines when feasible and with dashed lines in their respective colors when unfeasible. The boundary $f_C\,(k_t,\,k_{t+1}) = 0$, instead, is represented by a black dashed line. In both cases the effectiveness of public and private expenditure in the health production function is equally shared (i.e. $\alpha = 0.5$), while the share of the public budget allocated to the health system is either a low or high value (i.e. $\gamma = 0.5$ and $\gamma = 0.9$). Notice that any positive initial condition generates a trajectory that converges to k^*_A, the feasible steady state for all choices of γ, as long as $\alpha = 0.5$.

When the effectiveness of public and private expenditure in the health production function is equally shared, the feasible steady state is characterized by positive private healthcare

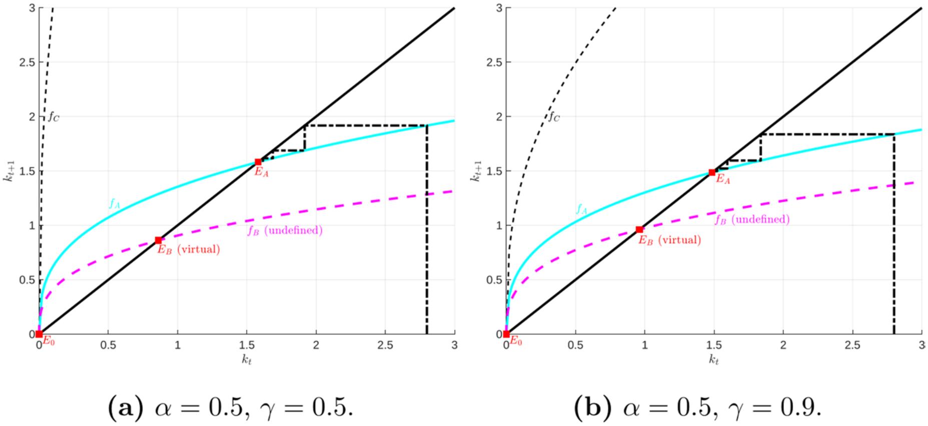


Fig. 2 Koneig–Lamerary staircase diagram for f_A

by the elderly, regardless of the level of public budget allocated to the health system. In this scenario, the value by the elderly place on private healthcare expenditure exceeds that of public healthcare expenditure for any value of γ. As a result, the elderly will always choose to allocate a portion of their savings to healthcare spending, independent of the public budget commitment to healthcare.

3.2 Scenario B: no private healthcare spending

We now consider the case where Proposition 3.1 point (b) is verified. This corresponds to combinations of α and γ that lie within the region shaded in magenta in Fig. 1b. In this case, R_B as defined by equation (40), applies, and the evolution of capital per worker is described by f_B as given in equation (38).

In the following Proposition 3.3, we show that f_B has an explicit form and we outline its key properties.

Proposition 3.3 *The implicit function f_B in equation (38) is equivalent to the following explicit function $\hat{f}_B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$:*

$$\hat{f}_B(k_t) = \frac{\beta p_t(k_t)(1-\tau)A(1-\delta)k_t^{\delta}}{1+\beta p_t(k_t)}. \tag{41}$$

Moreover, this function has the following properties:
(a) *It passes through the origin.*
(b) *It is strictly monotonically increasing.*
(c) $\lim_{k_t \to +\infty} \hat{f}_B(k_t) = +\infty.$
(d) $\lim_{k_t \to 0^+} \hat{f}'_B(k_t) = +\infty.$

Proof See Appendix C. □

Observe also that, although it cannot be proven analytically, numerous numerical experiments have shown that $\hat{f}_B$ is concave. By considering the above mentioned considerations, the following remark trivially holds.

Remark 3.4 The function $\hat{f}_B$ admits a unique positive fixed point $E_B = (k_B^*, k_B^*)$ which is asymptotically stable. If $E_B \in R_B$, then it represents a feasible fixed point; otherwise it is a virtual fixed point.

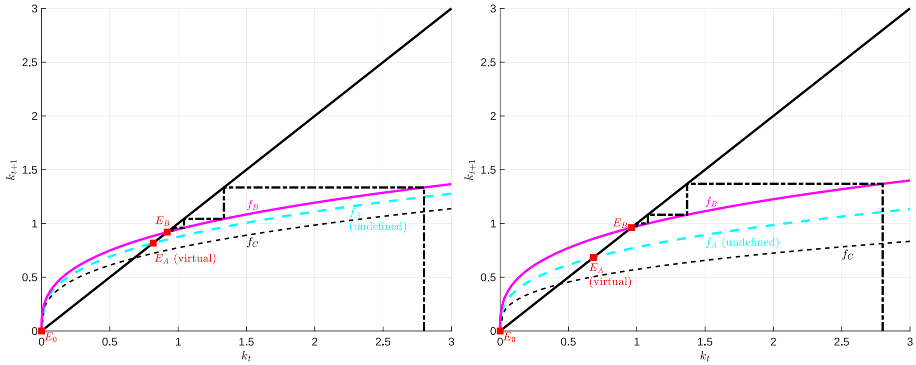
The main evidence is represented graphically in Fig. 3, showing the scale diagrams in which the generic trajectory converges to k_B^*. Notice that this scenario emerges if α and γ are large enough.

In this case, while high effectiveness of public spending in producing “health" is a necessary condition for transitioning from an equilibrium with private healthcare spending by the elderly to one with only public healthcare spending, it is not sufficient on its own. In fact, when public spending is more effective than private spending, and provided the public budget allocated to the health system is sufficient large, the model admits a unique positive steady state that is locally asymptotically stable and characterized by no private healthcare spending by individuals.

This outcome arises because, although public healthcare spending is highly effective in generating health, this alone is does not make it optimal for the elderly to forgo private healthcare expenditures. For this shift to occur, public healthcare spending-despite its higher productivity compared to private spending-must account for a substantial portion of overall public revenues. Under these conditions, the system benefits from a large and highly efficient public healthcare sector, making it no longer advantageous for the elderly to allocate their savings to private healthcare.

3.3 Switching between scenarios

As shown, both scenarios, with or without private healthcare spending, may arise depending on the parameter values. More precisely, both f_A and f_B admit, at most one unique positive fixed point (denoted k_A^* and k_B^*, respectively), which can be either feasible (if it lies within R_A or R_B) or virtual (if it does not belong to the domain of the corresponding function).



(a) $\alpha = 0.8$ and $\gamma = 0.7$.

(b) $\alpha = 0.8$ and $\gamma = 0.9$.

Fig. 3 Koneig–Lamerary staircase diagram for f_B

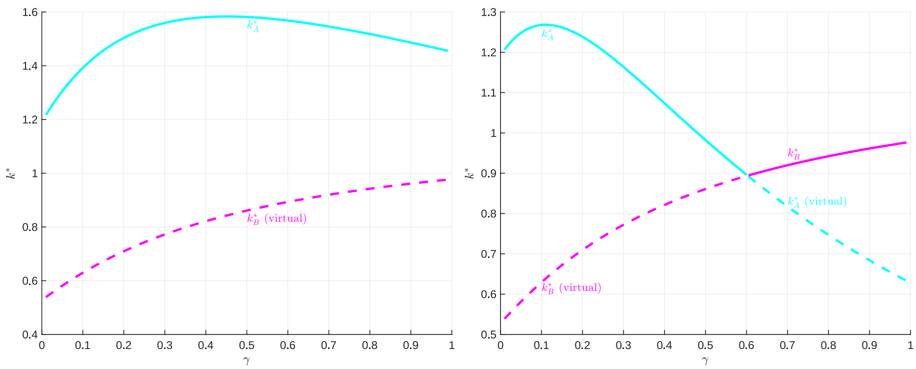
A key question, however, concerns how the system transitions between these scenarios as a parameter -specifically, the share of the public budget allocated to the healthcare system- changes.

We first consider the case with low α-value, i.e. $\alpha = 0.5$ so that, as emerges from Fig. 1b, the fixed point always belong to R_A, i.e., a positive fraction of savings is allocated to private healthcare for all levels of public budget devoted to health system γ. The resulting equilibrium value of capital per worker, $k^* = k_A^*$ as moving $\gamma \in (0, 1)$, is shown in Fig. 4a. The virtual fixed point associated with the no-private-spending scenario typically results in a lower level of k^*, assuming the feasible equilibrium solution is maximizing. However the corresponding curve (in cyan) exhibits a hump-shaped pattern: as γ increases, the equilibrium value of capital per worker initially rises and then declines. This suggests the existence of an optimal public budget share for healthcare that maximizes the equilibrium outcome.

We now turn to the case of a high α-value, specifically $\alpha = 0.8$. In contrast to the previous case, the resulting scenario here depends on the γ-value, the share of the public budget allocated to the health system. For low values of γ, the model again produces scenario A, characterized by positive healthcare spending. In this case, the maximum capital per young worker level is reached at $\gamma \simeq 0.11$. However, as the share of the public spending on healthcare increases and crosses the threshold value $\gamma \simeq 0.6$, a transition occurs: the fixed point k_A^* becomes virtual, while the fixed point k_B^* becomes feasible. This marks a shift to a new situation in which there is no private healthcare spending.

In this case, since public healthcare spending is highly effective, then there exists a threshold level of tax revenue allocated to healthcare that shifts the equilibrium from Case A to Case B. This implies that, if public spending is highly productive in generating “health", there is no need for the elderly to allocate part of their savings to health expenditure. In these countries, an effective and extensive public healthcare system alone is sufficient to ensure a good health status. Conversely, if public and private healthcare spending are equally effective, only the scenario involving private health expenditure can emerge. In this case, no level of public spending is sufficient to prevent the elderly from investing in “health".

Finally, we aim to asses whether, and to what extent, the optimal choices emerging from both scenarios affect both life quantity and health status.



(a) $\alpha = 0.5$. **(b)** $\alpha = 0.8$.

Fig. 4 The equilibrium point k^* for region A (in cyan) and region B (in magenta) being $\gamma \in (0,\ 1)$

Longevity is given by $p_t(k_t)$ as defined in equation (35), while health status in old age -depending on the scenario- is described by the following equations:

$$h^A_{t+1} = \frac{\left\{\alpha\gamma\tau k^\delta_{t+1}\left[\bar{p} + \frac{(1-\bar{p})\gamma\tau A(1-\delta)k^\delta_t}{1+\gamma\tau A(1-\delta)k^\delta_t}\right] + (1-\alpha)(1-\tau)\left(1 + A\delta k^{\delta-1}_{t+1}\right)k^\delta_t\right\} A(1-\delta)\theta}{1 + (\beta+\theta)\left[\bar{p} + \frac{(1-\bar{p})\gamma\tau A(1-\delta)k^\delta_t}{1+\gamma\tau A(1-\delta)k^\delta_t}\right]}, \quad (42)$$

in the case of positive private healthcare spending, and

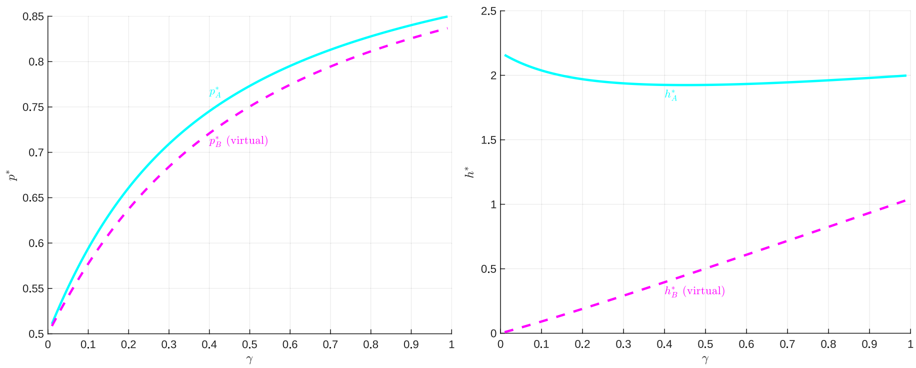
$$h^B_{t+1} = \alpha\gamma\tau A(1-\delta)k^\delta_{t+1}, \quad (43)$$

in the case of no private healthcare spending.

With regard to longevity, even if its functional form remains the same across both scenarios, it varies with the equilibrium level of capital per worker, which in turn depends on the share of the public budget γ allocated to the healthcare system.

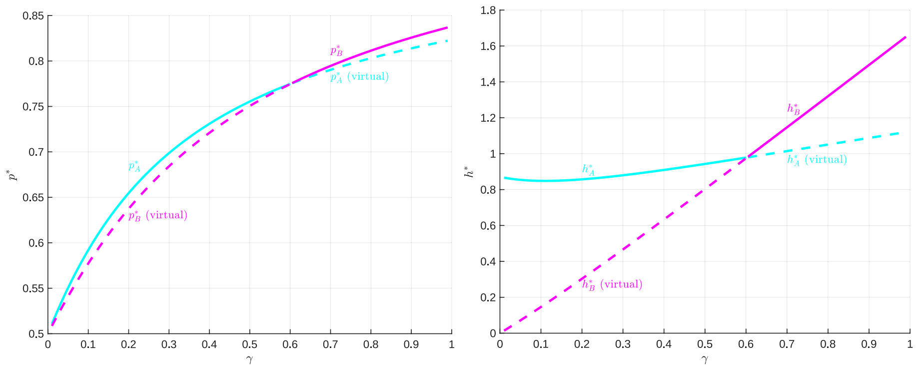
As shown clearly in both Figs. 5a and 6a, life expectancy increases as γ increases, even in the absence of private healthcare spending. This evidence holds regardless of whether public spending is relatively effective or not and it persists even when transitions between scenarios occur. This is because longevity is determined solely by public healthcare spending, not private expenditure, which makes the outcome similar in both scenarios.

However, a different behavior emerges when considering health status. Specifically, when the effectiveness of public healthcare spending is low, increasing the share of the public budget dedicated to health does not necessarily improve health status- this is evident in Fig. 5b. In contrast, when public spending is highly effective, even a shift between scenarios, does not hinder improvements: increasing the public budget share dedicated to health becomes an effective tool for enhancing the equilibrium level of health status (see Fig. 6b). Indeed, when public healthcare spending is significantly more productive than private spending, a scenario with only public healthcare -without any private health expenditure-may result in a higher health status. This provides important policy insights: if public healthcare spending is significantly more effective than private alternatives, and the State



(a) $\alpha = 0.5$. **(b)** $\alpha = 0.5$.

Fig. 5 Equilibrium values of p^* (panel a) and h^* (panel b) with $\alpha = 0.5$ for region A (in cyan) and region B (in magenta) being $\gamma \in (0, 1)$



(a) $\alpha = 0.8.$ **(b)** $\alpha = 0.8.$

Fig. 6 Equilibrium values of p^* (panel a) and h^* (panel b) with $\alpha = 0.8$ for region A (in cyan) and region B (in magenta) being $\gamma \in (0,\ 1)$

chooses to allocate a large share of its tax revenue to healthcare, a fully public system can ensure a higher health status than a mixed public-private model.

4 Conclusion

We considered an overlapping generations model to investigate the role of public versus private healthcare in determining both the quantity of life and health status in old age, as well as its impact on steady-state economic growth.

Depending on the comparison between the value of private and public health expenditures, two scenarios can emerge: either the elderly allocate a positive amount of their savings to private healthcare, or they choose not to invest in private health at all.

With regard to the existence and stability of steady-state equilibria in capital per worker - and the corresponding dynamics of life quantity and health status - we combine analytical tools with numerical methods to demonstrate that, in each scenario, a positive, locally asymptotically stable steady-state emerges. This steady state can be either feasible or virtual, depending on the model's parameter values.

Accordingly, we fixed the values of the main parameters and vary both the share of the public budget allocated to healthcare and the efficiency of public spending in the health production function.

On the one hand, our work is able to show that if public and private spending are equally effective in the health production function, then the feasible steady-state is characterized by positive private health expenditure by the elderly, regardless of the share of the public budget allocated to the health system. Differently, if public spending is more efficient than private expenditure, then there is a transition from a situation with positive private healthcare spending to a one with no private healthcare spending, as long as the share of the public budget devoted to the healthcare system is increased.

On the other hand, it shows that an increase in the share of the public budget allocated to the healthcare system has a positive effect on longevity in both scenarios, whether or not the elderly engage in private healthcare spending. However, the impact of this increase on health status may be ambiguous, as it depends critically on the effectiveness of public spending within the health production function. Specifically, when public expenditure is relatively ineffective, increasing the allocation of public resources to healthcare does not necessarily result in improvements in health status. In contrast, when public spending is highly effective, a larger allocation of public resources becomes a powerful tool to improve health status in old age.

The abovementioned results results suggest that policymakers could improve the effectiveness of public health expenditures by prioritizing preventive care programs, improving access to primary healthcare services, and investing in vaccination and disease control initiatives. In addition, targeted subsidies for essential treatments and the expansion of public hospital capacity could help ensure that increased public spending is translated into tangible improvements in both longevity and health outcomes.

4.1 Further developments of the model can be considered in future research

First, different health technologies could be considered that take into account the imperfect substitutability between public and private healthcare expenditures. In fact, considering imperfect substitutability, such as by adopting a Cobb-Douglas specification, the rate of technological substitution between public and private expenditure would no longer remain constant but would diminish. This means, for example, that as public healthcare expenditure increases, its marginal 'productivity' would fall relative to private expenditure. This adjustment could alter the optimal allocation between public and private spending; however, analyzing this case would require a new model, which may not be analytically tractable given the added complexity introduced by non-linearities.

Furthermore, the assumption of additivity in the utility function implies that the utility derived from consumption is independent of the individual's health status, and vice versa. The additive specification allows consumption and health-related decisions to be analyzed separately. Under this assumption, agents allocate savings to health solely based on its direct contribution to utility, without accounting for the way health might influence the marginal utility of consumption. In contrast, a non-additive (e.g., multiplicative) utility specification introduces complementarity between health and consumption. In such a framework, better health increases the marginal utility of consumption, thereby increasing the likelihood of higher investment in healthcare. Similarly, greater consumption can reinforce health and improve its marginal utility. While this approach is more realistic, it significantly increases the model's complexity and eliminates the possibility of treating the two choices independently. However, it represents a promising and natural extension, which we plan to pursue in future research.

Finally, an important extension of the model in future research could be to consider that longevity, as well as health in old age, depends not only on public spending but also on private spending. This inclusion could perhaps make the model difficult to study analytically, but it would certainly add interesting elements for further reflection.

Appendix A

We begin by making preliminary substitutions to simplify the maximization problem, reducing the number of constraints. To do this, we substitute the expression for g_{t+1} given by (9) into (13), and the expression for $\hat{R}_{t+1}$ given by (12) into (11). Then, from (11), we calculate s_t as

$$s_t = \frac{p_t}{R_{t+1}} \left(c_{t+1} + x_{t+1} \right) \tag{A.1}$$

and substitute this expression into (10). Finally, we substitute (13) into (8) and obtain the following equivalent model:

$$\max \quad U_t = \ln c_t + \beta p_t \ln c_{t+1} + \theta p_t \ln \left[\alpha\gamma\tau w_{t+1} + (1-\alpha) x_{t+1} \right] \tag{A.2}$$

$$\text{s.t.:} \quad c_t + \frac{p_t}{R_{t+1}} \left(c_{t+1} + x_{t+1} \right) = (1-\tau) w_t \tag{A.3}$$

$$-x_{t+1} \leq 0. \tag{A.4}$$

We solve model (A.2–A.4) using the method of Lagrange multipliers. In this regard, we define η_t as the Lagrange multiplier associated with constraint (A.3), and ξ_t as the Lagrange multiplier associated with constraint (A.4). We observe that model (A.2–A.4) is a mixed constrained maximization problem, as it includes both an equality constraint and an inequality constraint. According to, the theorem on Lagrange multipliers (see, for instance, Simon and Blume 1994, pp. 434 and 435), the constraint qualification requires that the rank of the Jacobian matrix -evaluated at a local maximizer and corresponding to the set of equality constraints and binding inequality constraints- be maximal. In our case, this matrix, regardless of the point at which it is evaluated, is given by:

$$\begin{bmatrix} 1 & \frac{p_t}{R_{t+1}} & \frac{p_t}{R_{t+1}} \\ 0 & 0 & -1 \end{bmatrix}.$$

Since its rank is always maximal, the constraint qualification condition is satisfied.

We can therefore construct the Lagrangian function, whose expression is:

$$\begin{aligned} \mathcal{L}_t \left(c_t,\, c_{t+1},\, \eta_t,\, \xi_t \right) = & \ln c_t + \beta p_t \ln c_{t+1} + \theta p_t \ln \left[\alpha\gamma\tau w_{t+1} + (1-\alpha) x_{t+1} \right] \\ & - \eta_t \left[c_t + \frac{p_t}{R_{t+1}} \left(c_{t+1} + x_{t+1} \right) - (1-\tau) w_t \right] + \xi_t x_{t+1} \end{aligned}$$

By differentiating the Lagrangian with respect to the decision variables c_t, c_{t+1}, and x_{t+1} we respectively obtain:

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = 0 : \quad \frac{1}{c_t} - \eta_t = 0 \tag{A.5}$$

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+1}} = 0: \quad \frac{\beta p_t}{c_{t+1}} - \eta_t \frac{p_t}{R_{t+1}} = 0 \tag{A.6}$$

$$\frac{\partial \mathcal{L}_t}{\partial x_{t+1}} = 0: \quad \frac{(1-\alpha)\,\theta p_t}{\alpha\gamma\tau w_{t+1} + (1-\alpha)x_{t+1}} - \eta_t \frac{p_t}{R_{t+1}} + \xi_t = 0. \tag{A.7}$$

It is also necessary to satisfy the initial constraints:

$$c_t + \frac{p_t}{R_{t+1}}\,(c_{t+1} + x_{t+1}) = (1-\tau)w_t \tag{A.8}$$

and

$$x_{t+1} \geq 0. \tag{A.9}$$

Moreover, the complementary slackness condition requires that:

$$\xi_t x_{t+1} = 0. \tag{A.10}$$

Finally, the non-negativity condition for the multiplier associated with the inequality constraint is given by:

$$\xi_t \geq 0. \tag{A.11}$$

From (A.5) we get

$$c_t = \frac{1}{\eta_t}. \tag{A.12}$$

Likewise, from (A.6) we get

$$c_{t+1} = \frac{\beta R_{t+1}}{\eta_t}. \tag{A.13}$$

We first consider the case when $x_{t+1} > 0$. We refer to this case as Case A. In light of (A.10), this implies that $\xi_t = 0$. By replacing $\xi_t = 0$ into (A.7), after some algebra, we find the following expression for x_{t+1}:

$$x_{t+1} = \frac{\theta}{\eta_t} R_{t+1} - \frac{\alpha}{1-\alpha}\gamma\tau w_{t+1}. \tag{A.14}$$

By substituting (A.12), (A.13), and (A.14) into (A.3), we find

$$\eta_t = \frac{1}{\frac{1}{1+(\beta+\theta)p_t}\left[\frac{p_t}{R_{t+1}}\frac{\alpha}{1-\alpha}\gamma\tau w_{t+1} + (1-\tau)w_t\right]}. \tag{A.15}$$

By substituting this value into (A.12), (A.13), and (A.14), we obtain the value of c_t, c_{t+1}, and x_{t+1} respectively. Finally, by imposing $x_{t+1} > 0$, we get the following condition:

$$(1-\alpha)\theta R_{t+1}(1-\tau)w_t > (1+\beta p_t)\alpha\gamma\tau w_{t+1}. \tag{A.16}$$

Case B arises when $\xi_t > 0$, which implies $x_{t+1} = 0$. From (A.7), we obtain

$$\xi_t = \frac{p_t}{R_{t+1}}\eta_t - \frac{(1-\alpha)\,\theta p_t}{\alpha\gamma\tau w_{t+1} + (1-\alpha)x_{t+1}}. \tag{A.17}$$

By substituting (A.12), (A.13), and $x_{t+1} = 0$ into (A.3), we find

$$\eta_t = \frac{1+\beta p_t}{(1-\tau)w_t}. \tag{A.18}$$

By substituting this value into (A.12) and (A.13), we obtain the value of c_t and c_{t+1}, respectively.

Moreover, by substituting (A.18) into (A.17), we find the condition characterizing Case B. The case when $\xi_t = 0$ and $x_{t+1} = 0$ represents the boundary between the two regions corresponding to Cases A and B. This justifies the use of $\geq$ (instead of $>$) in the inequality describing Case A.

The expressions (19) and (24) for s_t are obtained by substituting the values of c_{t+1} and x_{t+1} into (A.1), respectively for Cases A and B.

Finally, the Hessian matrix of the Lagrangian function with respect to the decision variables c_t, c_{t+1}, and x_{t+1} is given by:

$$H_{\mathcal{L}_t} = \begin{bmatrix} -\frac{1}{c_t^2} & 0 & 0 \\ 0 & -\frac{\beta p_t}{c_{t+1}^2} & 0 \\ 0 & 0 & -\frac{\theta(1-\alpha)^2 p_t}{[\alpha\gamma\tau w_{t+1}+(1-\alpha)x_{t+1}]^2} \end{bmatrix}. \tag{A.19}$$

This matrix is clearly negative definite. Consequently, the first-order conditions are also sufficient.

Appendix B

From (31), it follows that

$$w_{t+1} = A(1-\delta)k_{t+1}^{\delta}. \tag{B.1}$$

Likewise, (32) implies

$$r_{t+1} = A\delta k_{t+1}^{\delta}. \tag{B.2}$$

Consequently, from (33) we obtain:

$$R_{t+1} = 1 + r_{t+1} = 1 + A\delta k_{t+1}^{\delta}. \tag{B.3}$$

From (29) and (31), we can obtain a new expression for p_t as follows:

$$\bar{p} + \frac{(1-\bar{p})\gamma\tau w_t}{1+\gamma\tau w_t} = \bar{p} + \frac{(1-\bar{p})\gamma\tau A(1-\delta)k_t^{\delta}}{1+\gamma\tau A(1-\delta)k_t^{\delta}} =: p_t(k_t). \tag{B.4}$$

As stated in Proposition 2.1, the boundary separating regions A and B is given by the following equation:

$$\theta R_{t+1}(1-\tau)w_t = (1+\beta p_t)\frac{\alpha}{1-\alpha}\gamma\tau w_{t+1}. \tag{B.5}$$

Bringing all terms to the left-hand side, applying the previously mentioned substitutions, and making the dependence of p_t on k_t explicit as stated in (B.4), we define the left-hand side of this new equation as $f_C\,(k_t,\,k_{t+1})$. This allow us to express the boundary condition in terms of the equation

$$\left[\theta\left(1+A\delta k_{t+1}^{\delta-1}\right)(1-\tau)k_t^{\delta} - (1+\beta p_t(k_t))\frac{\alpha}{1-\alpha}\gamma\tau k_{t+1}^{\delta}\right]A(1-\delta) = 0.$$

Next, for both Cases A and Case B, we substitute the expression for s_t from (34), taking into account the previous substitutions both in the expression of s_t and in the conditions characterizing each case. From this, the thesis follows. As with the boundary condition, all terms must be brought to the left-hand side. We then define f_A for region A and f_B for region B.

Appendix C

(a) It is trivial to verify that $\hat{f}_B(0) = 0$.

(b) By applying the chain rule to compute $\hat{f}'_B(k_t)$, we obtain:

$$\hat{f}'_B(k_t) = \frac{\beta(1-\tau)A(1-\delta)}{(1+\beta\,p_t(k_t))^2}\left[p'_t(k_t)\,k_t^{\delta} + \delta\,p_t(k_t)\,k_t^{\delta-1}\,(1+\beta\,p_t(k_t))\right]. \tag{C.1}$$

Since all the parameters are positive, $k_t \geq 0$, $p_t(k_t) > 0$ for all $k_t \geq 0$, and $p'(k_t) > 0$ for all $k_t \geq 0$, it follows that $\hat{f}'_B(k_t) \geq 0$ for all $k_t \geq 0$. (c) Exploiting the fact that for all $k_t \geq 0$ it holds that $\bar{p} \leq p_t\,(k_t) < 1$, and we obtain:

$$\hat{f}_B\,(k_t) > \frac{\beta\bar{p}(1-\tau)A(1-\delta)}{1+\beta}k_t^{\delta}. \tag{C.2}$$

Applying the squeeze theorem, the claim follows. (d) We compute the first derivative at 0 by taking the limit of the incremental ratio as $h \to 0^+$. Specifically, also in light of (C.2), we have:

$$\lim_{k_t \to 0^+} \hat{f}_B'(k_t) = \lim_{h \to 0^+} \frac{\hat{f}_B(h) - \hat{f}_B(0)}{h} = \lim_{h \to 0^+} \frac{\hat{f}_B(h)}{h} > \lim_{h \to 0^+} \frac{\beta \bar{p}(1-\tau)A(1-\delta)}{1+\beta} \frac{1}{h^{1-\delta}} = +\infty.$$

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Declarations

Conflict of interest We declare that there are no conflict of interest.

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