On a property of the Vesica Piscis

Mauro Maria Baldi
mauro.baldi@unimib.it
Department of Computer Science, University of Milano-Bicocca, Milan, Italy

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Abstract

Vesica Piscis is a figure generated intersecting two circles having the same radius and such that the center of each circle lies on the exterior circumference of the other circle. The origin of the Vesica Piscis is ancient and unknown and this symbol has been present in all the ages up to nowadays. It has been proved that if two segments joining particular points of the Vesica Piscis are traced, they form a cross and their ratio is equal to the square root of three. One of these segments joins the intersecting points of the two exterior circumferences and the other segment joins the intersecting points between the segment joining the two centers of the two circles with the exterior circumferences. In this article, we prove that the converse is also true: if we consider two intersecting circles and the aforementioned segments such that their ratio is equal to the square root of three, then the two circles form a Vesica Piscis. We prove this results through two alternative proofs. The first proof uses Euclidean geometry and the second proof uses analytic geometry.

Keywords: Vesica Piscis, Sacred Geometry, Euclidean Geometry, Analytic Geometry.

1 Introduction

Vesica Piscis is a symbol consisting of the intersection of two circles having the same radius and where the center of one circle lies on the exterior circumference of the other circle (Figure 1). Sometimes the Vesica Piscis is also called *mandorla*, an italian word meaning almond, as the shape of the symbol recalls. Due to the shape that also resembles the female genital organ, before Christianity Vesica Piscis was a symbol that recalled the idea of generation as well as creation.

With early Christianity, the symbol was rotated ninety degrees and tails were added in such a way to recall the shape of a fish (Figure 2). In fact, in Latin Vesica Piscis means the bladder of the fish. One of the uses of this symbol

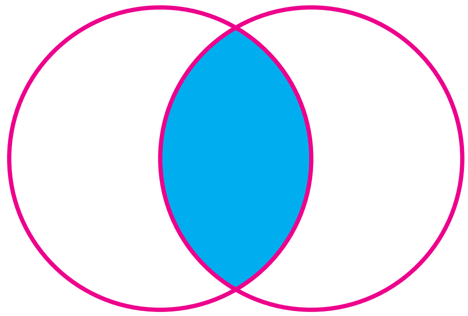

Figure 1: Vesica Piscis is the area highlighted in cyan subtended by the two circumferences.

was a coded way of communicating among Christians, in order to hide from the persecuting Romans. The Greek word *ichthys* ($\iota\chi\theta\acute{\upsilon}\varsigma$) means fish, but it is also an acronym for the words $I\eta\sigma o\tilde{\upsilon}\varsigma$ $X\rho\iota\sigma\tau\acute{o}\varsigma$ $\Theta\epsilon o\tilde{\upsilon}$ $Y\iota\acute{o}\varsigma$ $\Sigma\omega\tau\acute{\eta}\rho$, i.e. Jesus Christ Son of God and Savior. The symbol was drawn with chalk on the walls of temporary sites where the Christians met and it was used to indicate their presence to other Christians in a hidden manner.

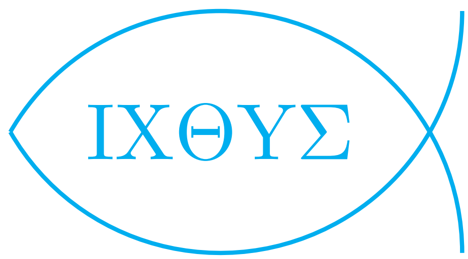

Figure 2: The hidden meaning of Christ in the Vesica Piscis.

Still in Christian culture, Vesica Piscis appears in paintings depicting Christ (Figure 3), in church facades (Figures 4 and 5), as well as at Saint Peter's Square (Figure 6).

But the Vesica Piscis hides deeper meanings: as shown in many works such as [Fletcher, 2004, Skinner, 2006, Lawlor, 2007, Sparavigna and Baldi, 2016a], if one traces the segment joining the centers of the two circles and another segment joining the intersection points of the exterior circumferences of the two circles, then a cross is obtained. The authors of these works prove that the ratio of these two segments is equal to the square root of three and this leads to finding the cross and the concept of trinity in a symbol heavily used in early Christianity [Lawlor, 2007, Fletcher, 2004]. As Skinner [2006] states, the square root of three was approximated by Pythagoras as 153:265, and the number 153 corresponds precisely to the number of fishes caught by the apostle Peter as

Figure 3: Painting of Christ inside a Vesica Piscis.

Figure 4: A church with a sculpture of a Madonna with her child inside a Vesica Piscis.

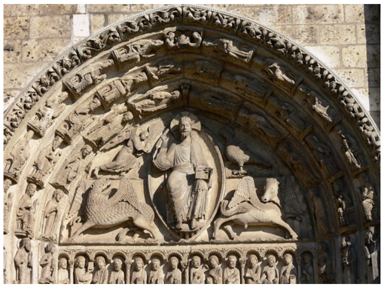

Figure 5: Christ inside a Vesica Piscis at the Chartres cathedral, France.

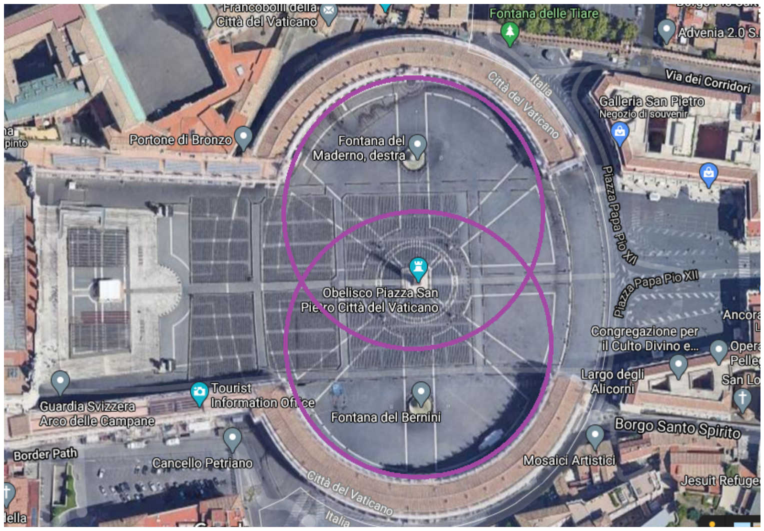

Figure 6: The Vesica Piscis at Saint Peter's square, Città del Vaticano.

narrated in the gospel (John 21, 11).

Moreover, the Vesica Piscis appears in Gothic and modern architecture, as well illustrated in [Barrallo et al., 2015], and recent studies focuse on a two-parameter family of shapes obtained by folding the Vesica Piscis [Mundilova and Wills, 2018]. Nowadays, the Vesica Piscis appears in many buildings and structures, like the Chalice well (Figure 7).

Finally, the Vescia Piscis is one of the symbols studied in Sacred Geometry, a branch of Geometry appearing in many settings such as in architecture and in nature [Hejazi, 2005]. In particular, the Vesica Piscis can be derived from another symbol: the Flower of Life [Sparavigna and Baldi, 2016b], from which regular polygons such as the pentagon [Sparavigna and Baldi, 2016c] can be derived.

The purpose of this article is to prove an additional property of the Vesica Piscis. As mentioned above, given a Vesica Piscis it is possible to demonstrate that the ratio between the two segments of the cross inside the *mandorla* is equal to the square root of three.

In this article, we also prove the opposite implication: given two circumferences incident in two points like in Figure 9, we consider two segments that naturally can be drawn: the segment joining the intersecting points of the two circumferences and the segment having as extreme points the points of intersection between the segment joining the two centers of the circumferences and the two circumferences themselves. We prove that if the ratio of these two segments is equal to the square root of three, then the two circles form a Vesica Piscis. In Section 2, we propose a proof using Euclidean geometry, while in Section 3 we propose an alternative proof using analytic geometry. We conclude in Section 4.

Figure 7: The Chalice well at Glastonbury Tor in the county of Somerset, England.

We provide a definition before stating and proving a theorem.

Definition 1 (cross ratio) *Given two circumferences intersecting in two distinct points of the plane, we define cross ratio the ratio between the length of the segment joining these two intersecting points and the length of the segment having as extreme points the intersection between the segment joining the centers of the two circumferences and the two circumferences.*

For instance, the cross ratio of the two circumferences depicted in Figure 8 is given by $\dfrac{\overline{PQ}}{\overline{AB}}$.

With this definition, we are now ready to prove the following:

Theorem 1 (Vesica Piscis) *Two intersecting circles subtend a Vesica Piscis if and only if their cross ratio is equal to* $\sqrt{3}$.

2 A proof using Euclidean geometry

$\Longrightarrow$ First implication: if two intersecting circumferences subtend a Vesica Piscis, then their cross ratio is equal to $\sqrt{3}$. We prove this implication in a similar fashion as done in Sparavigna and Baldi [2016a]. Without loss of generality, we can refer to Figure 8 for this part of the proof. Let $r > 0$ be the length of the radii of the two circumferences and let γ_1 and γ_2 respectively be the left and right circumferences in Figure 8. Still referring to Figure 8, we can say that $\overline{AB} = r$ because AB is a radius of both γ_1 and γ_2, and $\overline{AP} = r$ because AP is a radius of γ_1. Similarly, $\overline{BP} = r$ because BP is a radius of γ_2. Therefore,

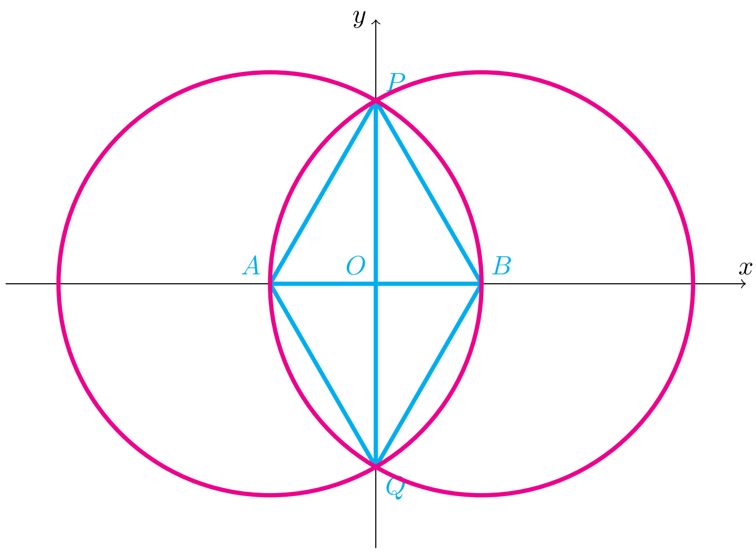

Figure 8: A geometrical construction for the first part of the proof.

triangle ABP is equilateral and so $\overline{PO} = \frac{\sqrt{3}}{2}\,\overline{AP} = \frac{\sqrt{3}}{2}r$. A similar conclusion can be stated if triangle ABQ is considered, leading to $\overline{OQ} = \frac{\sqrt{3}}{2}r$. Thus, $\overline{PQ} = \overline{PO} + \overline{OQ} = \sqrt{3}r$. The cross ratio is then:

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\sqrt{3}r}{r} = \sqrt{3}. \tag{1}$$

Note that it is possible to simplify r in (1) because we assumed $r > 0$ and so we have that $r \neq 0$.

$\Longleftarrow$ Opposite implication: if the cross ratio of two intersecting circumferences is equal to $\sqrt{3}$, then the two circumferences subtend a Vesica Piscis.

Without loss of generality, we can refer to Figure 9 for this part of the proof.

We know that $\frac{\overline{PQ}}{\overline{RS}} = \sqrt{3}$, which can be written as $\frac{\frac{\overline{PQ}}{2}}{\overline{RS}} = \frac{\sqrt{3}}{2}$. But, by construction, $\frac{\overline{PQ}}{2} = \overline{PO}$. Thus, we have: $\frac{\overline{PO}}{\overline{RS}} = \frac{\sqrt{3}}{2}$, which mean that triangle RSP is equilateral. Let l be the length of each side of this triangle and let $h = \frac{\sqrt{3}}{2}l$ be its height. Moreover, let $r > 0$ be the length of the radii of the two circumferences and let $d = \overline{AR}$. Segment AS is a radius of the circumference on the left and it is also the sum of segments AR and RS. Thus,

$$r = d + l. \tag{2}$$

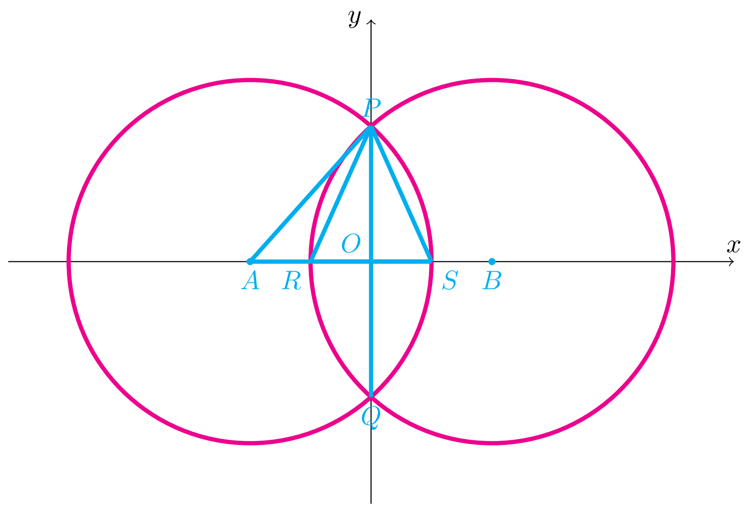

Figure 9: A geometrical construction for the second part of the proof.

By construction, triangle AOP is rectangle in O. If we apply Pythagora's theorem to this triangle, we get:

$$\left(d+\frac{l}{2}\right)^2+h^2=r^2. \tag{3}$$

But $h=\frac{\sqrt{3}}{2}l$ and so (3) reduces to:

$$\left(d+\frac{l}{2}\right)^2+\frac{3}{4}l^2=r^2. \tag{4}$$

If we put together (2) and (4), we get the following system of equations in the unknowns d and l:

$$\begin{cases} d+l=r \\ \left(d+\frac{l}{2}\right)^2+\frac{3}{4}l^2=r^2, \end{cases} \tag{5}$$

which can be solved as follows:

$$\begin{cases} d=r-l \\ \left(r-l+\frac{l}{2}\right)^2+\frac{3}{4}l^2=r^2; \end{cases} \tag{6}$$

$$\begin{cases} d=r-l \\ \left(r-\frac{l}{2}\right)^2+\frac{3}{4}l^2=r^2; \end{cases} \tag{7}$$

$$\begin{cases} d=r-l \\ r^2-rl+\frac{l^2}{4}+\frac{3}{4}l^2=r^2; \end{cases} \tag{8}$$

$$\begin{cases} d = r - l \\ rl = l^2. \end{cases} \tag{9}$$

Since the equilateral triangle RSP is non-degenerate, it follows that $l > 0$. Thus, we can divide both sides of the second equation in (9) by l and get $l = r$. Hence, we get the following solution:

$$\begin{cases} d = 0 \\ l = r. \end{cases} \tag{10}$$

Since $d = 0$, this implies that points A and R coincide and so the center of the circumference on the left in Figure 9 lies on the other circumference. The same considerations and computations can be done for the circumference on the right in Figure 9. Hence, the two circumferences subtend a Vesica Piscis.

3 A proof using analytic geometry

$\Longrightarrow$ First implication: if two intersecting circumferences subtend a Vesica Piscis, then their cross ratio is equal to $\sqrt{3}$.

Let $r > 0$ be the length of the radii of the two circumferences making up the Vesica Piscis. Without loss of generality, we can consider circumference γ_1 with center $A(-\frac{r}{2}, 0)$ and radius r and circumference γ_2 with center $B(\frac{r}{2}, 0)$ and radius r. We can compute the intersecting points of γ_1 and γ_2 as follows:

$$\begin{cases} \left(x + \frac{r}{2}\right)^2 + y^2 = r^2 \\ \left(x - \frac{r}{2}\right)^2 + y^2 = r^2; \end{cases} \tag{11}$$

$$\begin{cases} x^2 + rx + \frac{r^2}{4} + y^2 = r^2 \\ x^2 - rx + \frac{r^2}{4} + y^2 = r^2. \end{cases} \tag{12}$$

If we subtract side by side the second equation from the first equation in system (12), we get $2rx = 0$. Since $r > 0$, it must be $x = 0$ and system (12) becomes:

$$\begin{cases} x = 0 \\ y^2 = \frac{3}{4}r^2, \end{cases} \tag{13}$$

which solutions are points $P\left(0, \dfrac{\sqrt{3}}{2}r\right)$ and $Q\left(0, -\dfrac{\sqrt{3}}{2}r\right)$. The cross ratio is then:

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\left|\frac{\sqrt{3}}{2}r - \left(-\frac{\sqrt{3}}{2}r\right)\right|}{\left|\frac{r}{2} - \left(-\frac{r}{2}\right)\right|} = \sqrt{3}.$$

$\Longleftarrow$ Opposite implication: if the cross ratio of two intersecting circumferences is equal to $\sqrt{3}$, then the two circumferences subtend a Vesica Piscis.

Without loss of generality, given $h > 0$ we can consider points $R\left(-\frac{1}{\sqrt{3}}h,\, 0\right)$, $S\left(\frac{1}{\sqrt{3}}h,\, 0\right)$, $P\,(0,\, h)$ and $Q\,(0,\, -h)$. We have that

$$\frac{\overline{PQ}}{\overline{RS}} = \frac{|h-(-h)|}{\left|-\frac{1}{\sqrt{3}}h - \frac{1}{\sqrt{3}}h\right|} = \sqrt{3}. \tag{14}$$

Let γ_1 be the circumference passing through points P, Q and R and let γ_2 be the circumference passing through points P, Q and S. In light of (14), these circumferences have a cross ratio equal to $\sqrt{3}$. We then have to prove whether they form a Vesica Piscis. To do so, we first need to find their equations. It is known from analytic geometry that a circumference in a plane has equation $x^2+y^2+ax+by+c=0$. However, not all the equations in the aformentioned form represent a circumference. They do, provided they satisfy the condition $\frac{a^2}{4} + \frac{b^2}{4} - c \geq 0$. Note that if this condition is satisfied to equality, the circumference degenerates to a single point. If we impose the generic circumference $x^2+y^2+ax+by+c=0$ to pass through points P, Q and S, we find the following system of equations:

$$\begin{cases} h^2 + bh + c = 0 \\ h^2 - bh + c = 0 \\ \frac{1}{3}h^2 + \frac{1}{\sqrt{3}}ha + c = 0. \end{cases} \tag{15}$$

Summing up the first two equations in the system, we find $2h^2 + 2c = 0$ or $c = -h^2$, and the system becomes:

$$\begin{cases} c = -h^2 \\ h^2 - bh + c = 0 \\ \frac{1}{3}h^2 + \frac{1}{\sqrt{3}}ha + c = 0. \end{cases} \tag{16}$$

Plugging the first equation into the second and third ones, we have:

$$\begin{cases} c = -h^2 \\ -bh = 0 \\ \frac{1}{3}h^2 + \frac{1}{\sqrt{3}}ha - h^2 = 0. \end{cases} \tag{17}$$

Since we assumed $h > 0$, we can divide side by side by $-h$ and h respectively the second and third equation in the system and get the following solution:

$$\begin{cases} c = -h^2 \\ b = 0 \\ a = \frac{2}{\sqrt{3}}h. \end{cases} \tag{18}$$

Thus, the equation of circumference γ_1 is

$$x^2 + y^2 + \frac{2}{\sqrt{3}}hx - h^2 = 0. \tag{19}$$

We have that

$$\frac{a^2}{4} + \frac{b^2}{4} - c = \frac{4}{3}h^2 + h^2 \geq 0.$$

In particular, $\frac{4}{3}h^2 + h^2 > 0$ because we assumed $h > 0$. Therefore, (19) represents a non-degenerate circumference. Similar computations lead to the following equation for circumference γ_2:

$$x^2 + y^2 - \frac{2}{\sqrt{3}}hx - h^2 = 0. \tag{20}$$

Again, we have that

$$\frac{a^2}{4} + \frac{b^2}{4} - c = \frac{4}{3}h^2 + h^2 > 0.$$

Therefore, (20) represents a non-degenerate circumference too.

It is known from analytic geometry that if the equation $x^2+y^2+ax+by+c = 0$ represents a circumference, then its center has coordinates $\left(-\frac{a}{2}, -\frac{b}{2}\right)$. From (19) we find that the center of circumference γ_1 is point $\left(-\frac{1}{\sqrt{3}}h, 0\right)$, which coincides with point R and which, by construction, belongs to circumference γ_2. Moreover, from (20) we find that the center of circumference γ_2 is point $\left(\frac{1}{\sqrt{3}}h, 0\right)$, which coincides with point S and which, by construction, belongs to circumference γ_1. Thus, γ_1 and γ_2 subtend a Vesica Piscis.

4 Conclusions

In this paper, we revised the importance of the Vesica Piscis from ancient times to nowadays and we studied a property of the Vesica Piscis. After introducing the concept of cross ratio, we recalled that previous works proved that each Vesica Piscis has a cross ratio equal to the square root of three. We proved that the opposite implication is also true: if two circumferences are such that their cross ratio is equal to the square root of three, then they form a Vesica Piscis. This study allows us to conclude that two circumferences subtend a Vesica Piscis if and only if their cross ratio is equal to the square root of three.

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