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Elderly labor supply, endogenous grandparental childcare, and fertility in an OLG model

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ABSTRACT

This paper explores how grandparental childcare influences fertility decisions, elderly labor supply, and economic development. We model the time elderly individuals allocate between work and caring for grandchildren. In high-income countries, higher wages lead to increased investments in children's education and human capital. Interestingly, the decision to work in old age does not depend on economic development but on wage dynamics. Specifically, when the effective wage grows rapidly, grandparents are more likely to work, even while caring for grandchildren. These findings highlight the importance of intergenerational time transfers in shaping economic outcomes. Future research could explore the long-term effects of aging populations on labor markets and family dynamics.

1. Introduction

In its latest report on global aging, the United Nations noted that in 80 years, 24 percent of the world's population will be over 65 years old. Indeed, *In 2022 there were 771 million people aged 65 years or over globally, three times more than in 1980 (258 million). The elderly population is projected to reach 994 million by 2030 and 1.6 billion by 2050. As a result, by 2050 there will be more than twice as many individuals aged 65 or older globally as children under the age of five. Moreover, the number of persons aged 65 years or over globally will be almost the same as the number of children under age 12* (United Nations, 2022).

Increasing longevity has also increased the time that grandparents can devote to caring for their grandchildren. Specifically, retired parents might invest time in providing childcare and such intergenerational time transfers can significantly influence family decisions. However, limited attention has been given to the role of grandparents as childcare providers, although many families rely on grandparents' time to raise their children. For example, in the EU and the United States (US), 44% and 39% of grandparents provide care to their grandchildren respectively (Glaser et al., 2013; Livingston and Parker, 2010).

The effects of grandparent-provided childcare are multiple: they influence the labor supply and fertility choices of parents, as well as the labor supply of grandparents, ultimately impacting economic growth. (Hirazawa and Yakita, 2017) used an overlapping generations model (OLG) with endogenous labor supply of elderly people in a context where the elderly can allocate their (uncertain) time between work and leisure. The results revealed a positive relationship between per capita wage income and fertility. This positive relationship stems from the fact that as per capita income increases, so does life expectancy. This induces individuals to leave the labor market later, enabling them to earn more wage income and therefore increasing the number of children when young.

Regarding parents' labor supply, increased availability of grandparental childcare may encourage parents, particularly mothers, to enter the workforce or increase their work hours. With this in mind, Cardia and Ng (2003) calibrate an OLG model finding that grandparental time transfers increase the labor supply of the second generation. In an empirical analysis, Dimova and Wolff (2011) found that this phenomenon holds true for young European mothers. Similarly, Posadas and Vidal-Fernandez (2013) determined that grandparents' presence increases mothers' labor force participation in the US. Furthermore, Aparicio-Fenoll and Vidal-Fernandez (2015) demonstrated that the increased legal female retirement age implemented in Italy in 2000 led to a reduction in grandmothers' provision of childcare and decreased the labor force participation of their daughters. Finally, Fanti and Gori (2014) used an OLG model to investigate how exogenous provision of grandparental childcare activity inside the home affects the relationship between longevity and long-term economic growth. The results revealed that increased longevity may increase the labor supply.

Previous literature has produced substantial knowledge regarding the labor supply implications of transitioning into parenthood. While

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there is data available concerning the impact of grandparent-provided childcare on parents' labor participation, there is a noticeable lack of information regarding its effects on the labor supply of grandparents themselves.

Regarding the effects of grandparent-provided childcare on grandparental labor supply, the provision of grandparental childcare may affect the labor supply decisions of elderly individuals who act as caregivers. This is because they may choose to reduce their labor force participation or retire earlier to dedicate more time to caregiving responsibilities. This decision could lead to a decrease in the labor supply of the elderly.

Rupert and Zanella (2018) showed that, on average, employed women in the United States decrease their work commitment by approximately 30% when becoming grandmothers, compared with women who had not yet become grandmothers. Conversely, grandfathers did not exhibit a significant alteration in their labor participation. In another study, Frimmel et al. (2022) used Austrian data, finding that becoming a grandmother raised the likelihood of retiring from the workforce. Applying an instrumental variable approach, Backhaus and Barslund (2021) analyzed the causal relationship between grandparenthood and working-age grandparents' employment patterns across nine European nations. The results revealed a substantial adverse influence of grandparenthood on the workforce engagement of women aged 55 to 64. Notably, no significant was adjustment observed in male labor supply after becoming a grandparent.

The contemporary rise in life expectancy and declining fertility rates are the primary drivers of population aging, posing a potential threat to the sustainability of modern welfare states. In response to the anticipated surge in public spending on old-age pensions, many European nations have implemented reforms intended to raise the minimum retirement age in the past three decades (Hinrichs, 2021; Börsch-Supan and Coile, 2021). However, as Frattola (2023) stressed, while such reforms are intended to prolong the working lifespan, they may inadvertently decrease younger generations' fertility rates. Should adult children opt to postpone childbearing until their elderly parents retire, the upward adjustment of retirement age could further delay fertility decisions. In fact, numerous empirical studies have emphasized the significance of grandparents' childcare availability in shaping childbearing decisions. Approximately 50% of grandparents in the US and Europe contribute to childcare in some capacity (Thomese and Liefbroer, 2013). As a result, younger adults' fertility decisions may be influenced by expectations that their parents will offer childcare assistance upon retirement. Consequently, the likelihood and extent of potential parental support in childcare directly impact younger adults' perceived costs of raising children affecting fertility decisions.

Finally, a recent study by Cipriani and Fioroni (2024) used an OLG model to analyze how pension policies and child allowances interact with grandparental childcare.

This study investigates how the mechanism described above affects elderly individuals' labor supply, young individuals' fertility decisions and consequent economic development. In summary, when parents are deciding how many children to have, they consider the amount of childcare that is likely to be provided by their parents and optimize the number of children they have based on this factor. Once the optimal number of children is determined, adults allocate their parental childcare time in a way that can sufficiently meet the needs of raising their chosen number of children when combined with assumed grandparent support. This concept has been examined ed in the works of Boserup et al. (2016) and Miyazawa (2016).

Our OLG model enables us to consider the variables associated with young adults' decisions regarding how many children to raise and elderly adults' considerations of how to balance their time between work in old age and caring for grandchildren. Therefore, we extend the work of Fanti and Gori (2014) by considering an endogenous time allocation for the elderly through the choice of how much time to dedicate to work and childcare. This study represents a preliminary attempt to integrate into the same model the decisions regarding the number of children to have, how much to work in old age, and how much time to dedicate to grandchildren. These choices are evidently interconnected. The results reveal that an increase in the effective wage tends to boost educational investment in children's human capital in high-income economies, resulting in improved human capital over time. The individuals' choice to work when elderly does not depend on a country's economic development. This is because the decision to work or dedicate time to caring for grandchildren is influenced by wage dynamics in high- and low-income countries. Specifically, if the present value of the effective wage growth rate is high (indicating a lower utility weight for childcare in old age), adults will choose to work in their later years, regardless of the degree of economic development.

The remainder of this paper proceeds as follows. Section 2 outlines the OLG model setup, Section 3 presents the analytical and numerical results, and Section 4 concludes the paper.

2. Model setup

We consider a production economy that is populated by overlapping generations who live for three periods covering childhood, adulthood and old age. Time is discrete and indexed by $t = 0, 1, 2, \ldots ..n$, while the length of each period is normalized to one. In the first period, children are raided and educated. In the second period, adults work, raise and educate their children and save for consumption in their old age. In the third period, elderly individuals can work while spending part of their time caring for grandchildren.

The economy at time t consists of N_{t-1} old agents, N_t adult agents, and N_{t+1} young agents. Assuming that adults at time t raise n_t children, the number of adult agents at time $t+1$ evolves according to the following rule:

$$N_{t+1} = n_t N_t, \tag{1}$$

where n_t represents the number of children per adult at time t. Adults decide how much time to dedicate to raising and educating children and working in old age. Productivity in the work of the elderly depends on old agents' health status, which is subsequently related to life expectancy[1]. As noted by Bloom et al. (2014), increased life expectancy is associated with improved general health and reduced morbidity. Nevertheless, since we consider the elderly to have a unit of time in their old age (*i.e.* we do not ask about the elderly's life expectancy) and do not consider elderly health conditions, we assume in a general (but realistic) way that the elderly have a lower earning capacity than adults: consequently, the earning capacity during old age is given by $\gamma \in (0, 1)$.

The novelty introduced into our model is allowing the elderly to allocate the unit of time they have between continuing to work and raising their grandchildren. This assumption clearly implies that grandparents can contribute to raising their grandchildren, enabling parents to dedicate a smaller amount of their resources to raising their children. Grandparents' time indirectly affects adult childrens' (parents') fertility decisions through the income effect as it replaces parents' resources in raising the children. Grandparents' time transfers to their adult children (parents) are not influenced by the number of grandchildren. Regardless of the number of grandchildren, grandparents' childcare time is determined before adult children decide on the number of children they will have.

At the beginning of the second period, each adult has a human capital endowment h_t that is used to work and earn income from work. Adults allocate income between consumption, raising and educating children and save what remains for future consumption.

[1] For a detailed analysis of the role of longevity see Cipriani (2018), Cipriani and Fioroni (2021) and Chen et al. (2024).

Therefore, the budget constraint in adulthood is as follows:

$$w_t h_t(1 - n_t z + \mu_t) = c_{1,t} + s_t + n_t e_t,$$

where s_t is the agents' life-cycle saving, w_t is the wage rate for labor, so that $w_t h_t$ represents the wage rate for effective labor, $z \in (0, 1)$ denotes the child-raising cost per child, which is assumed, following (Hirazawa and Yakita, 2017), to be equal to a constant proportion of the wage income, and μ_t is the fraction of time that grandparents devote to raising all n_t grandchildren. Finally, e_t is the per-child educational expenditure. Given parents' income, the costs of raising children and providing for their education directs resources away from alternative uses (*i.e.* present and future consumption).

Elderly people choose labor supply during old age (considering the lower productivity during this period) and the time to be devoted to raising grandchildren.

The old-age budget constraint is obtained as follows:

$$r_{t+1} s_t + \gamma w_{t+1} h_t (1 - \mu_{t+1}) = c_{2,t+1},$$

where r_{t+1} is the interest factor in the period $t + 1$, w_{t+1} is the elderly labor wage rate and $\gamma \in (0, 1)$ is elderly individuals' earning ability.

Individuals' lifetime utility is assumed, as usual, to be represented by a log-linear utility function of consumption, quality (human capital) and quantity (number) of children, and the time dedicated to raising grandchildren in old age. Therefore, adults' utility is determined as follows:

$$U_t = \ln c_{1,t} + \rho(\ln c_{2,t+1} + \theta \ln \mu_{t+1}) + \beta \ln h_{t+1} + \lambda \ln n_t, \tag{2}$$

where $c_{1,t}$ is consumption during adulthood, $c_{2,t+1}$ denotes consumption during old age, n_t represents the number of children, and h_{t+1} indicates children's human capital and all variables are non-negative. We emphasize the influence of the variable $\mu_{t+1} \in [0, 1]$, which represents the fraction of available time that adults at time t decide to devote to raising grandchildren when they are elderly (time $t + 1$). Parameter $\rho \in (0, 1)$ represents the discount factor, $\lambda \in (0, 1)$ and $\beta \in (0, 1)$ respectively represent the weight attached to the quantity and quality of children and $\theta \in (0, 1)$ denotes the weight given to the time dedicated to raising grandchildren. Conversely, given the unit of time available, $(1 - \mu_{t+1})$ is the fraction of the available time that adults at time t decide to devote to working when old (*i.e.* at time $t + 1$).

The human capital accumulation process is determined as follows:

$$h_{t+1} = \pi(\epsilon + \phi e_t)^{\delta} \bar{h}_t^{1-\delta}, \tag{3}$$

where e_t is educational expenditure and $\bar{h}_t$ denotes the average stock of human capital in generation t representing the spillover effect from society (*i.e.* the fact that the learning process is more effective if a child interacts with more educated individuals). Note that $h_t = \bar{h}_t$, based on the assumption of identical individuals within a generation. The inclusion of the parameter $\epsilon > 0$ guarantees that human capital remains positive even if parents do not invest in their children's education (see *e.g.* De la Croix and Doepke (2004),[2]. Parameter $\pi > 0$ indicates the technology of production of human capital, $\delta \in (0, 1)$ indicates the influence of parents in educating their children, $(1 - \delta)$ is the productivity of the average level of human capital in the economy, and $\phi \in (0, 1]$ denotes the influence of resources devoted to education, increasing its effectiveness.

The constrained maximization problem is constructed as follows:

$$\max U_t(c_{1,t}, c_{2,t+1}, \mu_{t+1}, n_t, e_t) = \ln c_{1,t} + \rho(\ln c_{2,t+1} + \theta \ln \mu_{t+1}) + \beta \ln h_{t+1} + \lambda \ln n_t \tag{4}$$

$$\text{s. t.:} \quad w_t h_t(1 - n_t z + \mu_t) = c_{1,t} + s_t + n_t e_t \tag{5}$$

$$r_{t+1} s_t + \gamma w_{t+1} h_t (1 - \mu_{t+1}) = c_{2,t+1} \tag{6}$$

$$h_{t+1} = \pi \left(\epsilon + \phi e_t\right)^{\delta} h_t^{1-\delta} \tag{7}$$

$$0 \le \mu_{t+1} \le 1. \tag{8}$$

2.1. Constrained maximization solution

This section presents the first- and second-order conditions for model (4)–(8). The main results are presented in Theorem 2.1, with the proof provided in Appendix A.

Theorem 2.1. *Let $\lambda - \beta\delta > 0$. The solutions to the constrained maximization problem (4)–(7) are as follows:*

A if $w_t h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $\frac{\gamma w_{t+1}}{w_t r_{t+1}} > \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t)$ then

$$C_A: \begin{cases} c_{1,t} = \dfrac{\left[(1+\mu_t)w_t + \gamma \frac{w_{t+1}}{r_{t+1}}\right]h_t}{1+(1+\theta)\rho+\lambda} \\ c_{2,t+1} = \dfrac{\rho\left[(1+\mu_t)r_{t+1}w_t + \gamma w_{t+1}\right]h_t}{1+(1+\theta)\rho+\lambda} \\ \mu_{t+1} = \dfrac{\rho\theta}{\gamma}\dfrac{(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}}{1+(1+\theta)\rho+\lambda}\dfrac{r_{t+1}}{w_{t+1}} \\ e_t = \dfrac{\beta\delta}{\lambda-\beta\delta} z w_t h_t - \dfrac{\lambda}{\lambda-\beta\delta}\dfrac{\epsilon}{\phi} \\ n_t = \dfrac{\lambda-\beta\delta}{1+(1+\theta)\rho+\lambda}\dfrac{\left[(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}\right]h_t}{z w_t h_t - \frac{\epsilon}{\phi}} \\ s_t = \dfrac{1}{1+(1+\theta)\rho+\lambda}\left[(1+\theta)\rho(1+\mu_t)w_t - \gamma(1+\lambda)\dfrac{w_{t+1}}{r_{t+1}}\right]h_t \end{cases},$$

B if $w_t h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $\frac{\gamma w_{t+1}}{w_t r_{t+1}} \le \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t)$ then

$$C_B: \begin{cases} c_{1,t} = \dfrac{(1+\mu_t)w_t h_t}{1+\rho+\lambda} \\ c_{2,t+1} = \dfrac{\rho(1+\mu_t)r_{t+1}w_t h_t}{1+\rho+\lambda} \\ \mu_{t+1} = 1 \\ e_t = \dfrac{\beta\delta}{\lambda-\beta\delta} z w_t h_t - \dfrac{\lambda}{\lambda-\beta\delta}\dfrac{\epsilon}{\phi} \\ n_t = \dfrac{\lambda-\beta\delta}{1+\rho+\lambda}\dfrac{(1+\mu_t)w_t h_t}{z w_t h_t - \frac{\epsilon}{\phi}} \\ s_t = \dfrac{\rho}{1+\rho+\lambda}(1+\mu_t)w_t h_t \end{cases},$$

C if $\frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < w_t h_t \le \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $\frac{\gamma w_{t+1}}{w_t r_{t+1}} > \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t)$ then

$$C_C: \begin{cases} c_{1,t} = \dfrac{\left[(1+\mu_t)w_t + \gamma \frac{w_{t+1}}{r_{t+1}}\right]h_t}{1+(1+\theta)\rho+\lambda} \\ c_{2,t+1} = \dfrac{\rho\left[(1+\mu_t)r_{t+1}w_t + \gamma w_{t+1}\right]h_t}{1+(1+\theta)\rho+\lambda} \\ \mu_{t+1} = \dfrac{\rho\theta}{\gamma}\dfrac{(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}}{1+(1+\theta)\rho+\lambda}\dfrac{r_{t+1}}{w_{t+1}} \\ e_t = 0 \\ n_t = \dfrac{\lambda}{1+(1+\theta)\rho+\lambda}\left(1+\mu_t+\dfrac{\gamma}{r_{t+1}}\dfrac{w_{t+1}}{w_t}\right)\dfrac{1}{z} \\ s_t = \dfrac{1}{1+(1+\theta)\rho+\lambda}\left[(1+\theta)\rho(1+\mu_t)w_t - \gamma(1+\lambda)\dfrac{w_{t+1}}{r_{t+1}}\right]h_t \end{cases},$$

D if $\frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < w_t h_t \le \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $\frac{\gamma w_{t+1}}{w_t r_{t+1}} \le \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t)$ then

$$C_D: \begin{cases} c_{1,t} = \dfrac{(1+\mu_t)w_t h_t}{1+\rho+\lambda} \\ c_{2,t+1} = \dfrac{\rho(1+\mu_t)r_{t+1}w_t h_t}{1+\rho+\lambda} \\ \mu_{t+1} = 1 \\ e_t = 0 \\ n_t = \dfrac{\lambda}{1+\rho+\lambda}(1+\mu_t)\dfrac{1}{z} \\ s_t = \dfrac{\rho}{1+\rho+\lambda}(1+\mu_t)w_t h_t \end{cases}.$$

[2] As in Wang and Kimura (2024) this parameter can also consider the public education provided by the government.

Define R_A, R_B, R_C, and R_D as follows:

$$R_A = \left\{ (w_t, h_t, \mu_t, w_{t+1}, r_{t+1}) \in \mathbb{R}^5 : w_t h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta} \text{ and } \frac{\gamma w_{t+1}}{w_t r_{t+1}} > \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t) \right\},$$

$$R_B = \left\{ (w_t, h_t, \mu_t, w_{t+1}, r_{t+1}) \in \mathbb{R}^5 : w_t h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta} \text{ and } \frac{\gamma w_{t+1}}{w_t r_{t+1}} \le \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t) \right\},$$

$$R_C = \left\{ (w_t, h_t, \mu_t, w_{t+1}, r_{t+1}) \in \mathbb{R}^5 : \frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < w_t h_t \le \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta} \text{ and } \frac{\gamma w_{t+1}}{w_t r_{t+1}} > \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t) \right\},$$

$$R_D = \left\{ (w_t, h_t, \mu_t, w_{t+1}, r_{t+1}) \in \mathbb{R}^5 : \frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < w_t h_t \le \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta} \text{ and } \frac{\gamma w_{t+1}}{w_t r_{t+1}} \le \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t) \right\},$$

then, as long as $\lambda - \beta\delta > 0$, Theorem 2.1 establishes different solutions C_A, C_B, C_C, and C_D for the maximization problem in R_A, R_B, R_C, and R_D.[3]

Note that, from a purely mathematical perspective, the second-order conditions $\lambda - \beta\delta > 0$ arise from satisfying the requirement that the Hessian matrix must be negative definite on the set defined by equality and active inequality constraints. This condition may also have an economic interpretation. Recalling that β represents the significance parents assign to their children's human capital in the utility function, and δ denotes the productivity effect of education e_t in human capital production h_{t+1}, the term $\beta\delta$ captures the utility parents derive from the human capital generated through investment in children's education. In addition, λ is the significance attached to the number of children, where condition $\frac{\beta\delta}{\lambda} \in (0,1)$ indicates that the parental utility associated with having children is greater than the utility associated with human capital. This condition has been regularly used in literature (Yakita, 2010; De La Croix and Doepke, 2003).

According to Theorem 2.1, we examine four cases, namely A, B, C, and D.

Let us first observe that, with the exception of region A, the other regions include a variable between μ_{t+1} and e_t that takes on a boundary value. For instance, among other things, regions B and D are characterized by $\mu_{t+1} = 1$, whereas regions C and D are characterized by $e_t = 0$. From a mathematical perspective, the reason for these boundary values is attributable to the complementary slackness conditions that arise in solving the model. As illustrated in the proof of Theorem 2.1, these conditions are as follows:

$$\chi_t e_t = 0 \tag{9}$$

and

$$\xi_t \left(\mu_{t+1} - 1\right) = 0. \tag{10}$$

For example, cases A and B are characterized by $e_t > 0$; therefore, considering (9), we obtain $\chi_t = 0$.

Conversely, if $\chi_t > 0$, based on Eq. (9), it must be $e_t = 0$. This condition characterizes cases C and D. Similar considerations apply when considering (10).

Cases A and B represent developed economies with a high effective wages (high-income countries), while cases C and D refer to underdeveloped economies with low effective wages. In high-income countries (*i.e.* when $w_t h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$), adults invest in their children's education, whereas they do not in low-income countries, *i.e.* $e_t = 0$.

Note that an effective wage rate threshold exists at which an agent begins to invest in children's education. The economic logic underlying this behavior can be determined by comparing the marginal cost–benefit ratio of educating children with the marginal cost–benefit ratio of raising a child. Specifically, the marginal benefit of an additional child is given by $\frac{\lambda}{n_t}$, while the marginal cost is given by $w_t h_t z$ at $e_t = 0$. Regarding the marginal benefit/marginal cost ratio of educating children, it is given by $\frac{\beta\delta\phi}{\epsilon n_t}$. If the former ratio is smaller than the latter, then parents will prefer to invest in their children's education rather than have another child. In the contrasting case at which the effective wage rate is less than or equal to $\frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$, parents will prefer to have more children rather than invest their wages in educating the children.

We can also make a further distinction within rich and poor countries, depending whether adults (at time t) decide to work when they are old. In fact, in the cases in which this condition applies $\frac{\gamma w_{t+1}}{w_t r_{t+1}} > \frac{\rho\theta}{1+\rho+\lambda}(1+\mu_t)$, $\mu_{t+1} < 1$, that is, in cases A and C, adults find it worthwhile in old age to allocate part of their time to raising grandchildren and part of their time to working. This convenience depends on the effective wage growth rate in terms of the present value (*i.e.* $\frac{\gamma w_{t+1}}{w_t r_{t+1}}$). If the growth rate of the present value of the effective wage rate ($\frac{\gamma w_{t+1}}{w_t r_{t+1}}$) is high *i.e.* the utility weight of childcare during old age ($\rho\theta$) is low, adults will decide to work in old age. The economic logic of this inequality is the comparison between the marginal benefit of raising grandchildren (linked to deriving utility) and the opportunity cost of dedicating time to grandchildren, *i.e.* the effective wage they would obtain if they worked when elderly. This condition of equality between the marginal benefit and the marginal cost of raising grandchildren is represented by the partial derivative of the Lagrangian with respect to μ_{t+1} (Appendix A, Eq. (A.8)). From this optimal condition, substituting the other conditions, we obtain the inequality mentioned above.

We can refer to cases A and B (C and D) as developed economies (poor or developing) and to cases A and C (B and D) with high (low) growth rate of the present value of the effective wage as economies with positive elderly labor supply.

The positive effect of grandparents' childcare on parental fertility is evident. Specifically, the partial derivative $\frac{\partial n_t}{\partial \mu_t}$ is positive in all scenarios, indicating that if grandparents spend more time raising grandchildren, this reduces parents' opportunity cost of having children, subsequently increasing fertility.

2.2. Production sector

For the production sector, we assume the economy's aggregate production technology is depicted by a constant-returns-to-scale production function involving labor (L_t) and physical capital (K_t). We assume that capital stock fully depreciates after a single period of use; therefore, the capital stock in a given period equals the savings from the preceding one. We denote aggregate output by Y_t and represent, as usual, aggregate technology using the following Cobb–Douglas production function:

$$Y_t = A K_t^{\alpha} L_t^{1-\alpha}. \tag{11}$$

Assuming that the old agents work, the effective labor can be written as:

$$L_t = N_t h_t + N_{t-1}\gamma(1-\mu_t)h_{t-1} \tag{12}$$

or in per-young worker terms:

$$\frac{L_t}{N_t} = h_t + \frac{N_{t-1}}{N_t}\gamma(1-\mu_t)h_{t-1} \tag{13}$$

i.e.

$$\frac{L_t}{N_t} = h_t + \frac{\gamma(1-\mu_t)h_{t-1}}{n_{t-1}} = m_t. \tag{14}$$

[3] Partitioning of the solution space into four regions can also be found in Hirazawa and Yakita (2017).

The first-order conditions for profit maximization are obtained as follows:

$$w_t := \frac{A(1-\alpha)K_t^{\alpha}}{L_t{}^{\alpha}} \tag{15}$$

and

$$r_t := \frac{A\alpha K_t^{\alpha-1}}{L_t{}^{\alpha-1}}. \tag{16}$$

Define $k_t = \dfrac{K_t}{N_t}$. Therefore, formulae (15) and (16) become the following:

$$w_t = \frac{A(1-\alpha)k_t^{\alpha}}{m_t^{\alpha}} \tag{17}$$

and

$$r_t = A\alpha\frac{k_t^{\alpha-1}}{m_t^{\alpha-1}}. \tag{18}$$

Referencing Hirazawa and Yakita (2017), we also have the following:

$$k_{t+1} = \frac{s_t}{n_t}. \tag{19}$$

2.3. Market clearing conditions and dynamical systems

In this section, we determine the discrete time dynamical systems obtained by incorporating conditions from firms' profit maximization problem into the conditions of the constrained utility maximization problem. Since the first- and second-order conditions apply to four regions, we have four dynamic systems, one for each region. Proposition 2.2 illustrates these systems, and the proof is presented in Appendix B.

Proposition 2.2. *Let $\lambda-\beta\delta > 0$ and consider Theorem 2.1. The dynamical systems associated with the four regions are as follows:*

A if $A(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $(1+\mu_t)A\alpha\frac{\frac{k_t^{\alpha}}{m_t^{\alpha}}}{\frac{k_{t+1}}{m_{t+1}}} < \frac{1+\rho+\lambda}{\rho\theta}\gamma$

$$k_{t+1} = \frac{1}{\lambda-\beta\delta}\frac{(1+\theta)\rho(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} - \gamma(1+\lambda)\frac{k_{t+1}}{m_{t+1}}}{(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} + \gamma\frac{k_{t+1}}{m_{t+1}}}\left[zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}\right] \tag{20a}$$

$$m_{t+1} = \pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^{\delta}\left[zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}\right]^{\delta} h_t^{1-\delta} + \frac{1}{\lambda-\beta\delta}\frac{\alpha}{1-\alpha}\left[\gamma(1+\rho+\lambda)\frac{k_{t+1}}{m_{t+1}} - \rho\theta(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}}\right]\cdot \frac{zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}}{\left[(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} + \gamma\frac{k_{t+1}}{m_{t+1}}\right]\frac{k_{t+1}}{m_{t+1}}} \tag{20b}$$

$$\mu_{t+1} = \frac{\rho\theta}{\gamma}\frac{(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} + \gamma\frac{k_{t+1}}{m_{t+1}}}{[1+(1+\theta)\rho+\lambda]\frac{k_{t+1}}{m_{t+1}}} \tag{20c}$$

$$h_{t+1} = \pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^{\delta}\left[zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}\right]^{\delta} h_t^{1-\delta} \tag{20d}$$

B if $A(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $(1+\mu_t)A\alpha\frac{\frac{k_t^{\alpha}}{m_t^{\alpha}}}{\frac{k_{t+1}}{m_{t+1}}} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma$

$$k_{t+1} = \frac{\rho}{\lambda-\beta\delta}\left[zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}\right] \tag{21a}$$

$$m_{t+1} = \pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^{\delta}\left[zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}\right]^{\delta} h_t^{1-\delta} \tag{21b}$$

$$\mu_{t+1} = 1 \tag{21c}$$

$$h_{t+1} = \pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^{\delta}\left[zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t - \frac{\epsilon}{\phi}\right]^{\delta} h_t^{1-\delta} \tag{21d}$$

C if $\frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < A(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t \leq \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $(1+\mu_t)A\alpha\frac{\frac{k_t^{\alpha}}{m_t^{\alpha}}}{\frac{k_{t+1}}{m_{t+1}}} < \frac{1+\rho+\lambda}{\rho\theta}\gamma$

$$k_{t+1} = \frac{z}{\lambda}\frac{(1+\theta)\rho(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} - \gamma(1+\lambda)\frac{k_{t+1}}{m_{t+1}}}{(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} + \gamma\frac{k_{t+1}}{m_{t+1}}}A(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t \tag{22a}$$

$$m_{t+1} = \left\{\pi\epsilon^{\delta}h_t^{-\delta} + \frac{z}{\lambda}\frac{A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}}}{\frac{k_{t+1}}{m_{t+1}}}\frac{\gamma(1+\rho+\lambda)\frac{k_{t+1}}{m_{t+1}} - \rho\theta(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}}}{(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} + \gamma\frac{k_{t+1}}{m_{t+1}}}\right\}h_t \tag{22b}$$

$$\mu_{t+1} = \frac{\rho\theta}{\gamma}\frac{(1+\mu_t)A\alpha\frac{k_t^{\alpha}}{m_t^{\alpha}} + \gamma\frac{k_{t+1}}{m_{t+1}}}{[1+(1+\theta)\rho+\lambda]\frac{k_{t+1}}{m_{t+1}}} \tag{22c}$$

$$h_{t+1} = \pi\epsilon^{\delta}h_t^{1-\delta} \tag{22d}$$

D if $\frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < A(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t \leq \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $(1+\mu_t)A\alpha\frac{\frac{k_t^{\alpha}}{m_t^{\alpha}}}{\frac{k_{t+1}}{m_{t+1}}} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma$

$$k_{t+1} = \frac{\rho}{\lambda}zA(1-\alpha)\frac{k_t^{\alpha}}{m_t^{\alpha}}h_t \tag{23a}$$

$$m_{t+1} = \pi\epsilon^{\delta}h_t^{1-\delta} \tag{23b}$$

$$\mu_{t+1} = 1 \tag{23c}$$

$$h_{t+1} = \pi\epsilon^{\delta}h_t^{1-\delta} \tag{23d}$$

As noted previously, m_{t+1} represents the total labor supply given by the labor supply of young people and, where present, by the labor supply of old people. In cases B and D, where it is not worthwhile for the elderly to work, *i.e.* they use all their time caring for their grandchildren, the total labor supply m_{t+1} coincides with the labor supply of the young, *i.e.* h_{t+1}. However, in cases A and C, we obtain the labor supply of the elderly as the difference between m_{t+1} and h_{t+1}.

Referencing Hirazawa and Yakita (2017), we can construct a simpler formulation of the dynamical systems. It is easy to see that the ratios k_t/m_t are very common in the four formulations just stated. By defining the variable $x_t = k_t/m_t$, we can reformulate the aforementioned systems as illustrated in Proposition 2.3, with the proof presented in Appendix C. Notably, this variable (x_t) not only represents the capital–labor ratio but can also be read as income derived from wages since $w_t = \frac{A(1-\alpha)k_t^{\alpha}}{m_t^{\alpha}} = A(1-\alpha)x_t^{\alpha}$.

Proposition 2.3. *Let $\lambda-\beta\delta > 0$. The dynamical systems associated to the four regions from Theorem 2.1 can be reformulated as follows: (see the equations in Box I).*

A if $A(1-\alpha)x_t^{\alpha}h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $(1+\mu_t)A\alpha\frac{x_t^{\alpha}}{x_{t+1}} < \frac{1+\rho+\lambda}{\rho\theta}\gamma$

B if $A(1-\alpha)x_t^{\alpha}h_t > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ and $(1+\mu_t)A\alpha\frac{x_t^{\alpha}}{x_{t+1}} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma$

T_B:

$$x_{t+1} = \frac{\rho}{\pi}\frac{\left[zA(1-\alpha)x_t^{\alpha}h_t - \frac{\epsilon}{\phi}\right]^{1-\delta}}{(\beta\delta\phi)^{\delta}(\lambda-\beta\delta)^{1-\delta}h_t^{1-\delta}} \tag{25a}$$

$$\mu_{t+1} = 1 \tag{25b}$$

$$h_{t+1} = \pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^{\delta}\left[zA(1-\alpha)x_t^{\alpha}h_t - \frac{\epsilon}{\phi}\right]^{\delta} h_t^{1-\delta} \tag{25c}$$

$$T_A \begin{cases} x_{t+1} = \dfrac{\frac{1}{\lambda-\beta\delta}\left[(1+\theta)\rho(1+\mu_t)A\alpha x_t^\alpha - \gamma(1+\lambda)x_{t+1}\right]\left[zA(1-\alpha)x_t^\alpha h_t - \frac{\epsilon}{\phi}\right]^{1-\delta} x_{t+1}}{\pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^\delta \left[(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}\right] x_{t+1} h_t^{1-\delta} + \frac{1}{\lambda-\beta\delta}\frac{\alpha}{1-\alpha}\left[\gamma(1+\rho+\lambda)x_{t+1} - \rho\theta(1+\mu_t)A\alpha x_t^\alpha\right]\left[zA(1-\alpha)x_t^\alpha h_t - \frac{\epsilon}{\phi}\right]^{1-\delta}} & \text{(24a)} \\ \mu_{t+1} = \dfrac{\rho\theta}{\gamma}\dfrac{(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}}{\left[1+(1+\theta)\rho+\lambda\right]x_{t+1}} & \text{(24b)} \\ h_{t+1} = \pi\left(\dfrac{\beta\delta\phi}{\lambda-\beta\delta}\right)^\delta \left[zA(1-\alpha)x_t^\alpha h_t - \dfrac{\epsilon}{\phi}\right]^\delta h_t^{1-\delta} & \text{(24c)} \end{cases}$$

Box I.

C *if* $\frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < A(1-\alpha)x_t^\alpha h_t \leq \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ *and* $(1+\mu_t)A\alpha\frac{x_t^\alpha}{x_{t+1}} < \frac{1+\rho+\lambda}{\rho\theta}\gamma$

$$T_C \begin{cases} x_{t+1} = \dfrac{\frac{z}{\lambda}\frac{(1+\theta)\rho(1+\mu_t)A\alpha x_t^\alpha - \gamma(1+\lambda)x_{t+1}}{(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}} A(1-\alpha)x_t^\alpha}{\pi\epsilon^\delta h_t^{-\delta} + \frac{z}{\lambda}\frac{A\alpha x_t^\alpha}{x_{t+1}}\frac{\gamma(1+\rho+\lambda)x_{t+1} - \rho\theta(1+\mu_t)A\alpha x_t^\alpha}{(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}}} & \text{(26a)} \\ \mu_{t+1} = \dfrac{\rho\theta}{\gamma}\dfrac{(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}}{\left[1+(1+\theta)\rho+\lambda\right]x_{t+1}} & \text{(26b)} \\ h_{t+1} = \pi\epsilon^\delta h_t^{1-\delta} & \text{(26c)} \end{cases}$$

D *if* $\frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}} < A(1-\alpha)x_t^\alpha h_t \leq \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta}$ *and* $(1+\mu_t)A\alpha\frac{x_t^\alpha}{x_{t+1}} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma$

$$T_D \begin{cases} x_{t+1} = \dfrac{\rho}{\lambda\pi}\epsilon^{-\delta} zA(1-\alpha)x_t^\alpha h_t^\delta & \text{(27a)} \\ \mu_{t+1} = 1 & \text{(27b)} \\ h_{t+1} = \pi\epsilon^\delta h_t^{1-\delta} & \text{(27c)} \end{cases}$$

In addition to what was already demonstrated in Proposition 2.3, with a similar procedure to that outlined in Appendix C and starting from the first-order conditions, it is possible to express n_t in terms of the variables x_t, μ_t, and h_t. In particular, the following holds:

$$R_A: \quad n_t = \frac{\lambda-\beta\delta}{1+(1+\theta)\rho+\lambda}\frac{1-\alpha}{\alpha}\frac{\left[(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}\right]h_t}{zA(1-\alpha)x_t^\alpha h_t - \frac{\epsilon}{\phi}} \tag{28}$$

$$R_B: \quad n_t = \frac{\lambda-\beta\delta}{1+\rho+\lambda}\frac{(1+\mu_t)A(1-\alpha)x_t^\alpha h_t}{zA(1-\alpha)x_t^\alpha h_t - \frac{\epsilon}{\phi}} \tag{29}$$

$$R_C: \quad n_t = \frac{\lambda}{1+(1+\theta)\rho+\lambda}\left(1+\mu_t+\gamma\frac{x_{t+1}}{A\alpha x_t^\alpha}\right)\frac{1}{z} \tag{30}$$

$$R_D: \quad n_t = \frac{\lambda}{1+\rho+\lambda}(1+\mu_t)\frac{1}{z}. \tag{31}$$

Regarding the fertility rates in the different scenarios, as already noted, there is a positive contribution of the time grandparents dedicate to raising grandchildren (μ_t) with respect to fertility. The more time grandparents dedicate to raising grandchildren, the lower the expense for parents, *ceteris paribus*, of raising children. Furthermore, it is interesting to see the effect of x_t, *i.e.* youth wages, on fertility. The effect of an increase in young-age wage income on total fertility is negative, since an increase in wages, *ceteris paribus*, increases the cost of raising children and therefore reduces the fertility rate.

Systems T_A, T_B, T_C and T_D describe the evolution of the physical capital/labor ratio (x_t), childcare time (μ_t) and human capital (h_t) over time. Due to the complexity of the systems, in the following section we describe the main features of our model by combining analytical tools and numerical techniques.

3. Qualitative and quantitative dynamics

This section describes the evolution of the state variables over time. Unfortunately, the formulation of the dynamical systems in cases A and C is too complex to be approached analytically. In these cases, we use implicit maps that could be made explicit; however, their manageability would remain prohibitive, which makes it difficult (if not outright impossible) to examine properties such as the determination of fixed points and their nature from a purely theoretical perspective.

3.1. Elderly dedicating all their time to childcare

We begin by considering cases in which the growth rate of the present value of the effective wage is low, and adults plan to spend all their time in old age caring for their grandchildren (*i.e.* $\mu_{t+1} = 1$).

In Proposition 3.1, we report the results for cases B and D, where an explicit formulation of the respective dynamical systems enables us to achieve more results. The proof is presented in Appendix D.

Proposition 3.1. *Let the following conditions hold in region B:*

$$\frac{zA(1-\alpha)\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^\alpha}{zA(1-\alpha)\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^\alpha - \frac{\lambda-\beta\delta}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}} > \frac{\lambda}{\beta\delta} \tag{32}$$

and

$$2A\alpha\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^{\alpha-1} \geq \frac{1+\rho+\theta}{\rho\theta}\gamma, \tag{33}$$

then the point with the following coordinates:

$$E_B(x_B, \mu_B, h_B) = \left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}},\ 1,\ \frac{\epsilon}{\phi}\frac{1}{zA(1-\alpha)\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^\alpha - \frac{\lambda-\beta\delta}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}}\right)$$

denotes an equilibrium point for region B.

Moreover, let the following conditions hold for region D:

$$\sqrt{\frac{\lambda}{\beta\delta}} < \phi\pi^{\frac{1}{\delta}}\left[\left(\frac{\rho}{\lambda}\right)^\alpha zA(1-\alpha)\right]^{\frac{1}{1-\alpha}} \leq \frac{\lambda}{\beta\delta} \tag{34}$$

and

$$2\lambda\alpha\theta \geq z(1+\rho+\lambda)\gamma(1-\alpha), \tag{35}$$

then the point with the following coordinates:

$$E_D(x_D, \mu_D, h_D) = \left(\left[\frac{\rho}{\lambda}zA(1-\alpha)\right]^{\frac{1}{1-\alpha}},\ 1,\ \epsilon\pi^{\frac{1}{\delta}}\right)$$

is an asymptotically stable equilibrium point for region D.

The results of Proposition 3.1 also lend themselves to an interesting economic interpretation. In fact, we can argue that when grandparents dedicate all their time to raising their grandchildren ($\mu_{t+1} = 1$), then two unique equilibrium points exist for regions B and D. In particular, the equilibrium point E_D is asymptotically stable.

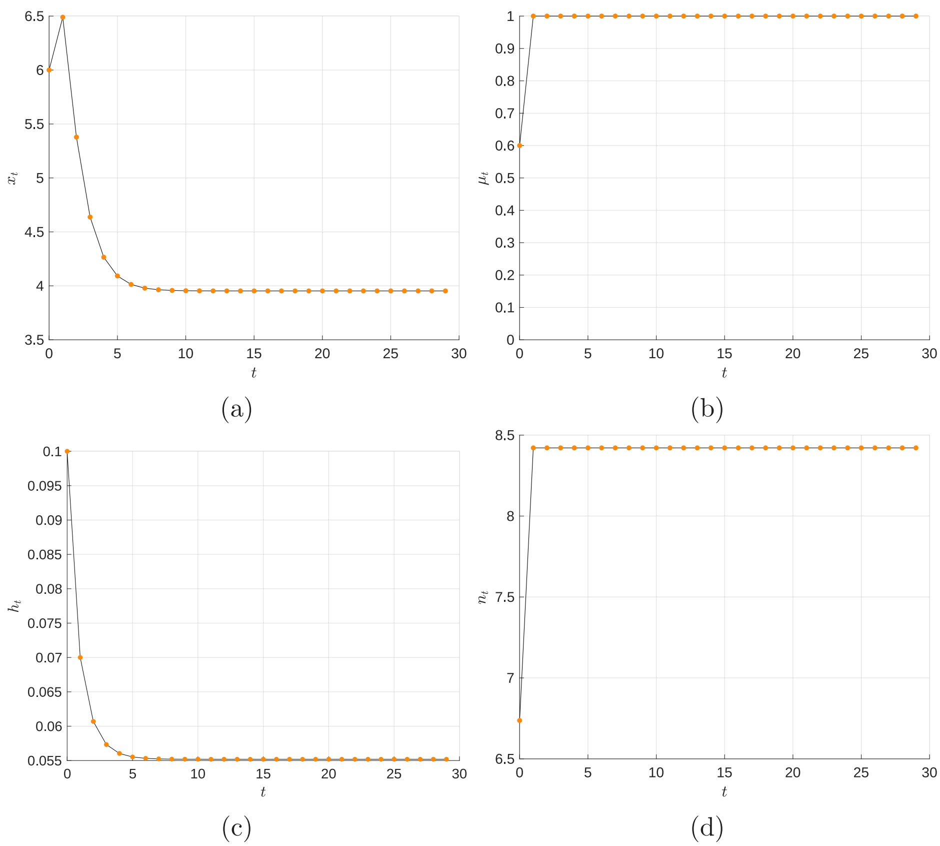

Fig. 1. Dynamics in case D for parameter values $A = 100$, $\alpha = 1/3$, $\beta = 0.1$, $\delta = 0.6$, $\epsilon = 0.1$, $\phi = 0.9$, $\gamma = 0.4$, $\lambda = 0.6$, $\rho = 0.3$, $\pi = 0.7$, $\theta = 0.3$, and $z = 0.075$ and initial conditions $x_0 = 6$, $\mu_0 = 0.6$, and $h_0 = 0.1$.

The long-term evolution of the model in cases in which all the time is planned to be dedicated to raising grandchildren in old age, can be better understood by showing the computational experiments.[4]

In the following Figures, we present four subplots related to the evolution of the state variables over time: in Panel (a), we will illustrate the dynamics of the physical capital/labor ratio x_t, in Panel (b), we will show the dynamics of the childcare time by the elderly μ_t, in Panel (c), we will demonstrate the dynamics of the human capital h_t and finally, in Panel (d), we will display the dynamics of fertility choice.

In Fig. 1, we present the trajectories in R_D (*i.e.* developing countries and low growth rate of the present value of the effective wage). As proved in Proposition 3.1, all variables converge to an asymptotically stable fixed point, while μ_t settles to 1 immediately after the initial time $t = 0$.

In this case, due to the low labor income, it is disadvantageous for adults to invest in educating their children as it leads to stabilizing human capital at lower levels. Additionally, the growth rate of the present value of the effective wage rate is low, making it unbeneficial for the elderly to work. Instead, it is better for the elderly to dedicate all their time to caring for their grandchildren. This contribution to child-rearing will promote a relatively high fertility rate.

Countries in Region D are characterized by low income, high fertility, older people retiring early to care for grandchildren and low education and human capital levels. This scenario can be found in regions with strong intergenerational family structures and informal economies, where older people are often more focused on caring for the family than continued employment. For example, Niger has a very low per capita income, an extremely high fertility rate (about 6.8 children per woman, the highest in the world), older people leaving the labor force early to support the family and low education levels. Another example is the Democratic Republic of Congo.

In Fig. 2 we present the trajectories of the dynamics entirely in region B (*i.e.* rich countries with low growth rate of the present value of the effective wage). Once again,0 μ_t immediately settles at 1 at $t = 1$, indicating no elderly labor supply.

In this case, high labor income enables adults to educate their children more easily, which leads to an increase in human capital over time. Unlike the previous case, this results in a growth in human capital, which suggests that the equilibrium point identified in Proposition 3.1 may either be unstable or have an attraction basin that does not align with region B. Furthermore, similar to the previous case, the growth rate of the present value of the effective wage rate remains low, making it unprofitable for the elderly to work. Instead, it is more beneficial for them to focus entirely on caring for their grandchildren, which contributes to a relatively high fertility rate.

This region corresponds to countries with high income, medium to high fertility, low employment rates among older workers and high human capital investment. These are generally advanced economies with generous welfare systems, family support and high-quality education. For example, France has high income, relatively high fertility (around 1.8 children per woman, which is one of the highest in Europe), older people leave the labor market at around 62–64 years of age and accessible universities, high-quality public education and strong vocational training support. A similar circumstance is observed in countries such as Denmark, Ireland and New Zealand (high income, medium-high fertility, low employment over 65 years of age and free

[4] We conducted the experiments using MATLAB, where the code considers the four regions. Considering the significant number of parameters, we explored several parameter configurations associating each region with a unique color. Specifically, region A is red, region B is magenta, region C is blue, and region D is orange. The MATLAB code for reproducing all the figures presented in this article is available in the associated repository (Baldi et al., 2025).

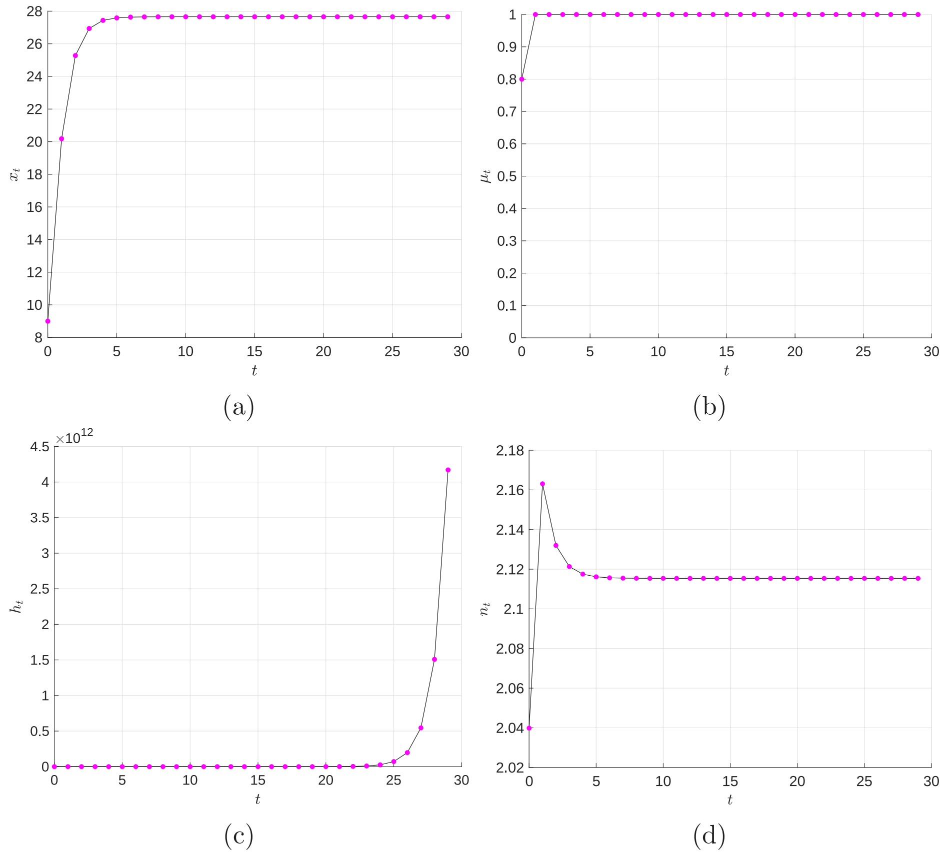

Fig. 2. Dynamics in case B for parameter values $A = 100$, $\alpha = 0.5$, $\beta = 0.5$, $\delta = 0.5$, $\epsilon = 1$, $\phi = 0.5$, $\gamma = 0.3$, $\lambda = 0.8$, $\rho = 0.8$, $\pi = 0.8$, $\theta = 0.5$, and $z = 0.2$ and initial conditions $x_0 = 9$, $\mu_0 = 0.8$, and $h_0 = 1$.

and innovative education systems with strong focus on digital skills and lifelong learning).

3.2. Elderly dedicating time to labor and childcare

As noted previously, a dynamical systems evaluation concerning regions A and C (*i.e.* characterized by the high growth rate of the present value of the effective wage) is too complex to be approached analytically.[5] Therefore, we only consider the open cases from a quantitative perspective.

In Fig. 3, we present the trajectories associated with the case belonging to region A in which young-age wage income is sufficiently high. While the values of x_t and μ_t converge to a given value, per-capita human capital h_t increases over time. This dynamic of human capital arises from the fact that parents in affluent countries find it worthwhile to invest resources on their children's education, which increases human capital. In this case, the elderly divide their time between caring for children and supplying labor, while the fertility rate is lower compared with previous cases.

Countries in Region A that combine high income, low fertility, an older population that remains in the labor market longer and strong human capital investment tend to be advanced economies with high-quality education and active skills development policies. A country that meet these criteria is Switzerland, which has one of the highest per-capita incomes in the world, a low fertility rate (around 1.4 children per woman) and an average labor market exit at around 65–66 years of age. Concerning human capital investment, Switzerland has an excellent education system, a high percentage of tertiary graduates and a high continuing education investment level. Similar arguments could be made for Sweden, Norway, Germany, the Netherlands and Japan.

The trajectories associated with region C are presented in Fig. 4 and refer to low values of young-age wage income. In contrast to the previous case, a stationary equilibrium is reached (*i.e.* h_t stabilizes as parents no longer consider it worthwhile to invest in their children's education, resulting in a decline in human capital). Additionally, similar to the previous case, the elderly labor supply remains positive and is associated with lower fertility rates of cases B and D.

Countries in region C with low income, older people still working, medium to low fertility and low education and human capital levels are often in developing countries with weak social systems and limited investment in education. For example, Bangladesh has low income, low to medium fertility, a high proportion of older people still working in the informal sector, low educational attainment and a high school dropout rate. India is also a low- to middle-income country with a declining fertility rate, many older people still working in agriculture and the informal sector, and low access to higher education.

3.3. Transition between regions

An interesting question is whether a switch between regions is possible (*i.e.* situations in which an elderly person's choices fluctuate over time by time, or where transitions between high and low growth rates of the present value of the effective wage can emerge). This question relates to the mathematical properties of the invariance of set R_i, $i \in \{A, B, C, D\}$. In the cases previously considered, a trajectory starting in a given region does not exit that region, meaning that the region is invariant. However, if a trajectory beginning in one region eventually enters a different region after a certain number of steps, the set is not invariant and transitions between regions are possible.

[5] Technically, it would be possible to move to an explicit form, however, the complexity would not be reduced and further qualitative studies cannot be done.

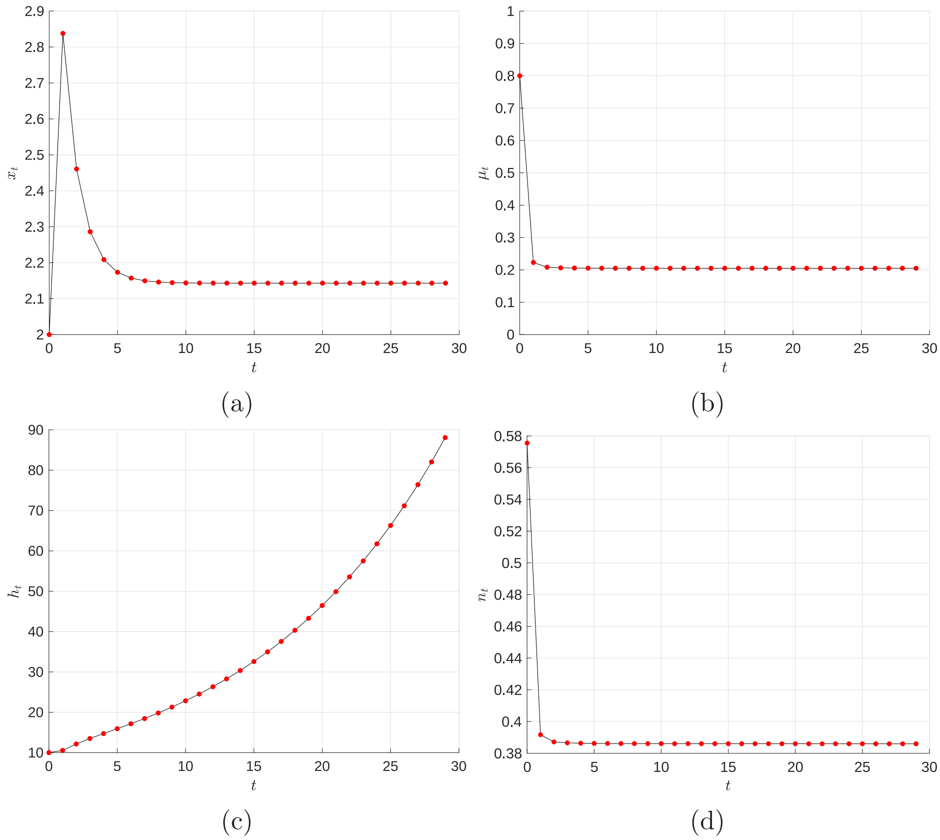

Fig. 3. Dynamics in case A for parameter values $A = 70$, $\alpha = 0.5$, $\beta = 0.5$, $\delta = 0.5$, $\epsilon = 0.1$, $\phi = 0.5$, $\gamma = 0.9$, $\lambda = 0.5$, $\rho = 0.1$, $\pi = 0.3$, $\theta = 0.1$, and $z = 0.5$ and initial conditions $x_0 = 2$, $\mu_0 = 0.8$, and $h_0 = 10$.

Considering the analytical complexity of our model, we can only approach this question by means of numerical simulations. Therefore, we can show a transition from one region to another employing numerical experiments.

One example is presented in Fig. 5, where trajectories starting in region A switch to region C, with each variable reaching a steady state. In this case, the elderly consistently benefit from dividing their between raising grandchildren and working – indicating a positive elderly labor supply – yet the economy transitions from a high-income to low-income circumstances.

Fig. 6 reveals similar and opposite situations in the transition from region C to A where x_t and μ_t stabilize and h_t diverges. This situation refers to the same case as Fig. 5, but the transition occurs in the opposite direction (*i.e.* the economy moves from a low to high wage income).

This result demonstrates that it is possible for trajectories to begin in region A and end in region C, but the reverse journey is also feasible. However, the latter case does not reach an equilibrium point but human capital levels rise. Furthermore, these results lead us to conclude that while there can be at least one asymptotically stable equilibrium point in region C, its basin of attraction does not coincide with the entire region.

The final outcome of our system depends on several factors, such as the parameter constellation and the initial conditions, meaning that a complete explanation of the motivations behind such behaviors is complex. However, some suggestions could be explored. In particular, Fig. 5 shows the case where, unlike in Fig. 3, both the human capital level and capital–labor ratio (as well as young-age wages) decrease over time. As a consequence, after several iterations, the product $h_t x_t$ becomes sufficiently low, the condition C_C on the effective wage is meet and the economy transitions from rich to poor. This mechanism could be linked to the value of λ, *i.e.* the utility weight of the number of children. In fact, it can be immediately observed that if λ is small enough (even close to zero) then the weight attached to fertility is low and education levels are positive, creating a virtuous mechanism with higher human capital. In this case, condition C_A cannot be violated and the economy remains rich. On the other hand, with higher λ, condition C_C on the effective wage could fail, no investment in education is made and the human capital level shows a decreasing trajectory. The opposite case can be explained as a sort of leverage effect that can emerge for low values of λ, and as a result economies may move from poor to rich.

When a country moves from region C to region A, the most noteworthy changes are the growth in per capita income, human capital and fertility. An example of a country that has made the transition from low income and high fertility to high income and low fertility is South Korea. South Korea went from being one of the poorest countries in the world in the 1960s, with a weak education system, high illiteracy, low human capital investment and high fertility (in the 1950s, South Korean women had an average of 5–6 children), to being an economic powerhouse today, with a high average gross domestic product per capita, one of the best education systems in the world, with a strong emphasis on science and technology, high human capital investment, with one of the highest graduation rates in the world and one of the lowest fertility rates in the world (around 0.8 children per woman in 2023).

In the other case we theoretically examine (*i.e.* the transition of a country from region A to region C), no possible examples exist in recent years as economic development and human capital tend to be enduring factors. However, there are some cases in which wars, economic crises, political instability and bad governance have resulted in drastic decline in economic and social conditions. For example, Venezuela was one of the richest countries in Latin America until the 1980s and 1990s, thanks to oil reserves, high education and health investment levels and an average fertility rate. Today, per-capita income is extremely low due

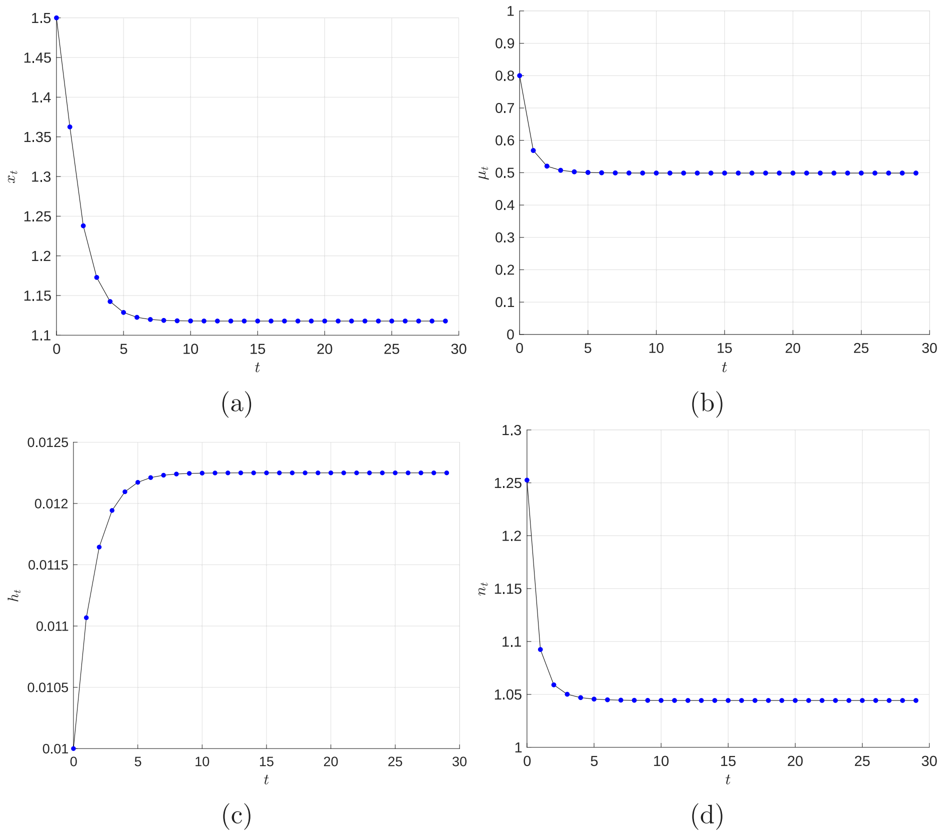

Fig. 4. Dynamics in case C for parameter values $A = 100$, $\alpha = 0.5$, $\beta = 0.4$, $\delta = 0.5$, $\epsilon = 0.1$, $\phi = 0.6$, $\gamma = 0.7$, $\lambda = 0.99$, $\rho = 0.05$, $\pi = 0.35$, $\theta = 0.2$, and $z = 0.7$ and initial conditions $x_0 = 1.5$, $\mu_0 = 0.8$, and $h_0 = 0.01$.

to poor governance, hyperinflation and corruption, and the education system has collapsed, resulting in a mass exodus of professionals and a rise in fertility among the poorer classes.

Finally, a different situation emerges in Fig. 7, switching from region B to region D. In both cases, the growth rate of the present value of the effective wage is sufficiently low, with the elderly primarily spending time caring for grandchildren. This causes a shift from high to low young-age income and the system stabilizes to the fixed point as shown in Proposition 3.1. Additionally, by combining the observations from Figs. 7 and 2, we conclude that both trajectories exist in region B. These are characterized by increasing human capital levels that remain in region B, while other trajectories overflow into region D, reaching the asymptotically stable equilibrium point.

3.4. Different initial states

This section examines the behavior of orbits under the same parameters but assuming different initial states. Specifically, we aim to infer whether changing the initial states preserves the behavior of the orbits through further graphical analyses. For example, based on what we previously illustrated, some orbits may remain entirely within region A and others may migrate from region A to region C. However, the two cases examined earlier differed in terms of parameter values. Nevertheless, from these observations, it is reasonable to assume that the initial state might influence the evolution of the orbit. In the additional experiments conducted, we encounter situations that either support or contradict this hypothesis.

For instance, using the same parameter configuration as in Fig. 2, Fig. 8 reveals that the orbits are entirely confined to region B. Specifically, the dynamics of x_t converge to the same equilibrium point, while those of h_t diverge. Although we are unable to provide a rigorous proof that the basin of attraction coincides with the entire region, after conducting additional tests beyond those in Fig. 8, we consistently observed convergence to the same value of x_t across various initial conditions. Therefore, it is logical to contend that the basin of attraction includes at least the entire region B (*i.e.* developed economies with low wage growth rate exhibit the same long-run dynamics), at least in this case.

We want to emphasize the words *at least* in the previous sentence. In fact, in other numerical tests, we find that the basin of attraction can also extend beyond its region of origin. For example, this is the case with the same parameter configuration as in Fig. 4, which pertains to region C. As clearly shown in Fig. 9, the dynamics of x_t corresponding to different initial values converge to a point within region C, but one of them begins in region A. Moreover, this result is consistent with the previously illustrated transition from region A to region C. In this case, the dynamics of h_t also converge. As a matter of fact, several numerical simulations show that the basin of attraction of the equilibrium point belonging to C can be larger than region C. Economies with high wage growth rates can transition from being developed to developing. Such a shift may occur if the product of the capital–labor ratio and per-capita human capital falls below a certain threshold, causing the economy to move from developed to developing countries.

In summary, the proposed experiments show that under the same parameter configuration, changing initial conditions may produce different outcomes. In some cases, we observe that orbits behave similarly within a neighborhood or across an entire region, while in other cases, the behavior varies depending on the initial conditions. Hence, economies can shift from one type to another assuming all other factors remain constant.

3.5. Equilibrium in region B

This section explores the equilibrium point in region B, as mentioned in Proposition 3.1. It is not immediately clear that a combination

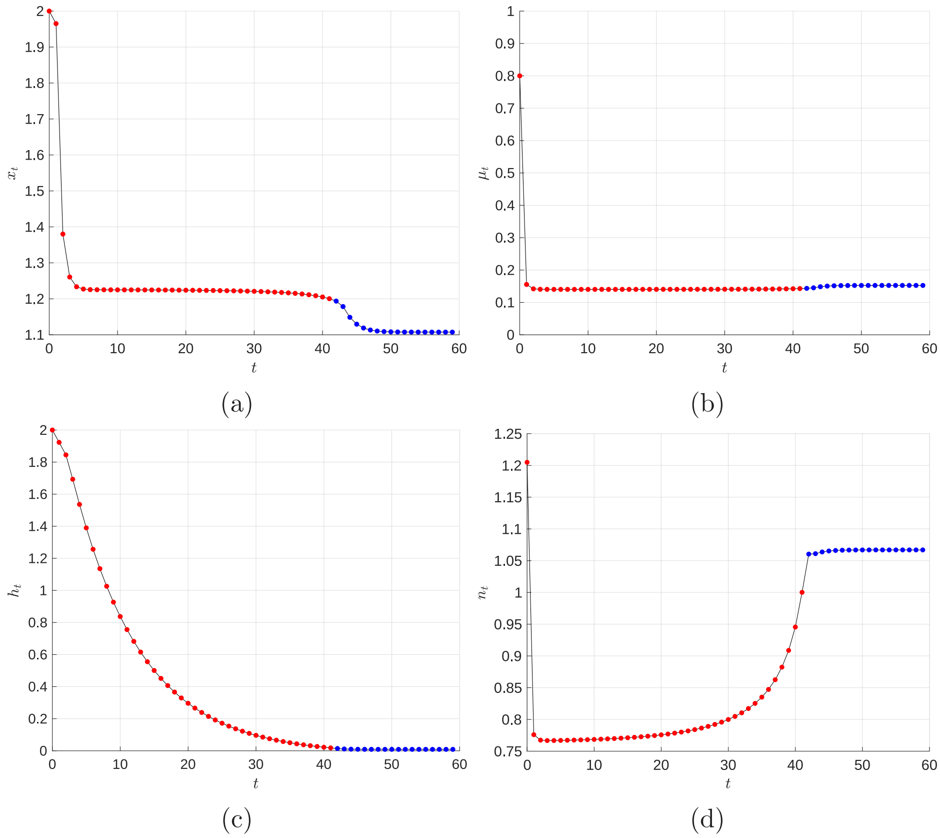

Fig. 5. Switching dynamics from case A to case C for parameter values $A = 100$, $\alpha = 0.25$, $\beta = 0.5$, $\delta = 0.5$, $\epsilon = 0.1$, $\phi = 0.6$, $\gamma = 0.9$, $\lambda = 0.9$, $\rho = 0.1$, $\pi = 0.3$, $\theta = 0.1$, and $z = 0.5$ and initial conditions $x_0 = 2$, $\mu_0 = 0.8$, and $h_0 = 2$.

of parameters exists that satisfies both conditions (32) and (33), and that is also located in region B. However, it has been shown that equilibrium point E_B exists when conditions (32) and (33) are met. For instance, the parameter combination associated with region B shown in Fig. 2 does not satisfy constraint (32). This naturally raises the question of whether there is a parameter combination satisfies both constraints (32) and (33), as well as the conditions defining region B. To investigate this, we used a solver to find a solution for the following nonlinear constrained optimization problem and determine whether a feasible solution exists:

$$\min \quad 1 \tag{36}$$

$$\text{s.t.:} \quad \lambda - \beta\delta > 0 \tag{37}$$

$$A(1-\alpha)x_0^{\alpha}h_0 > \frac{\epsilon}{\phi z}\frac{\lambda}{\beta\delta} \tag{38}$$

$$(1+\mu_0)A\alpha\frac{x_0^{\alpha}}{x_1} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma \tag{39}$$

$$x_0 \geq 0 \tag{40}$$

$$\mu_0 \geq 0 \tag{41}$$

$$\mu_0 \leq 1 \tag{42}$$

$$h_0 \geq 0 \tag{43}$$

$$x_1 = \frac{\rho}{\pi}\frac{\left[zA(1-\alpha)x_0^{\alpha}h_0 - \frac{\epsilon}{\phi}\right]^{1-\delta}}{(\beta\delta\phi)^{\delta}(\lambda-\beta\delta)^{1-\delta}h_0^{1-\delta}} \tag{44}$$

$$\mu_1 = 1 \tag{45}$$

$$h_1 = \pi\left(\frac{\beta\delta\phi}{\lambda-\beta\delta}\right)^{\delta}\left[zA(1-\alpha)x_0^{\alpha}h_0 - \frac{\epsilon}{\phi}\right]^{\delta}h_0^{1-\delta} \tag{46}$$

$$\frac{zA(1-\alpha)\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^{\alpha}}{zA(1-\alpha)\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^{\alpha} - \frac{\lambda-\beta\delta}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}} > \frac{\lambda}{\beta\delta} \tag{47}$$

$$2A\alpha\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^{\alpha-1} \geq \frac{1+\rho+\theta}{\rho\theta}\gamma \tag{48}$$

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \lambda, \rho,\ \theta,\ z \in (0,\ 1) \tag{49}$$

$$\phi \in (0,\ 1] \tag{50}$$

$$A, \epsilon,\ \pi > 0. \tag{51}$$

As is well known, optimization solvers do not support strict inequality constraints. However, these can be easily converted into $\geq$ or $\leq$ constraints using appropriate techniques. In our experiment, the solver successfully provides a feasible solution that met all constraints, demonstrating that at least one combination of parameters exists in region B that ensures the existence of equilibrium point E_B. We next approximate the values provided by the solver to two decimal places and rechecked that these values satisfied all constraints (37)–(51). As this check was successful, we examine the equilibrium point with the following combination of parameters and initial values:

$A = 50.03$, $\alpha = 0.68$, $\beta = 0.58$, $\delta = 0.79$, $\epsilon = 2.23$, $\phi = 0.46$, $\gamma = 0.65$, $\lambda = 0.79$, $\pi = 0.51$, $\rho = 0.21$, $\theta = 0.52$, $z = 0.13$, $x_0 = 3.45$, $\mu_0 = 0.58$, and $h_0 = 3.38$.

This combination of parameters produces the following (approximated) values for the coordinates of E_B:

$x_B = 2.336527$, $\mu_B = 1.000000$, and $h_B = 330.121685$.

In Fig. 10, we present the dynamics for the proposed combination of parameters and initial point $(x_0, \mu_0, h_0) = (3.45, 0.58, 338)$, as well

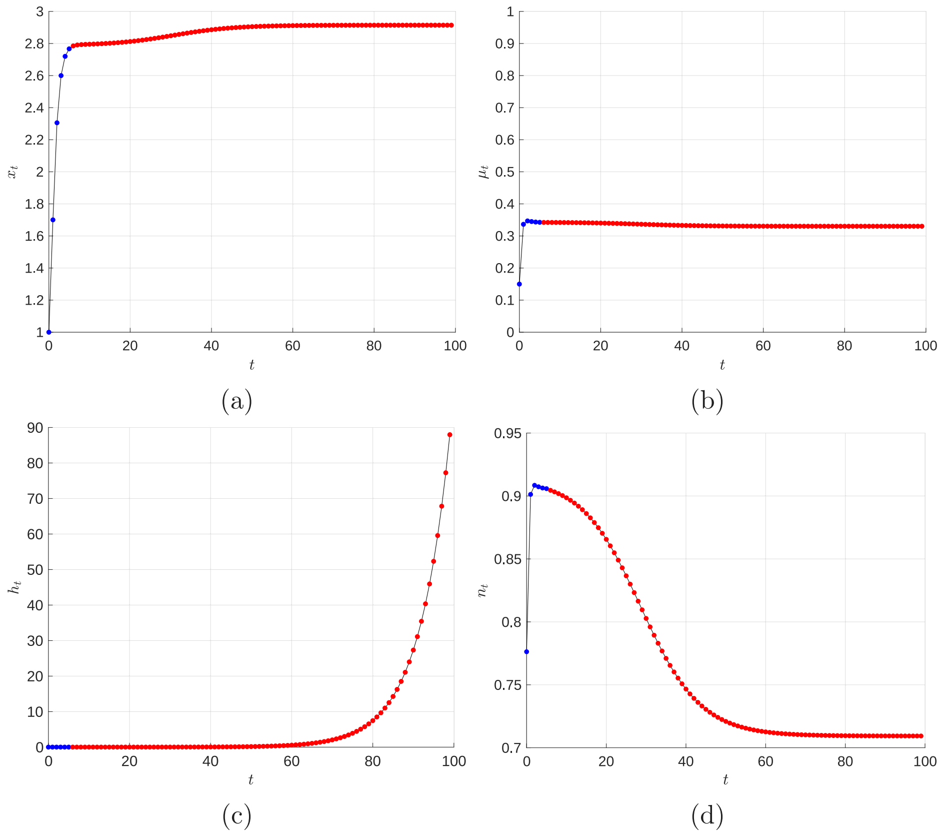

Fig. 6. Switching dynamics from case C to case A for parameter values $A = 100$, $\alpha = 0.35$, $\beta = 0.4$, $\delta = 0.5$, $\epsilon = 0.1$, $\phi = 0.6$, $\gamma = 0.7$, $\lambda = 0.95$, $\rho = 0.1$, $\pi = 0.35$, $\theta = 0.2$, and $z = 0.7$ and initial conditions $x_0 = 1$, $\mu_0 = 0.15$, and $h_0 = 0.01$.

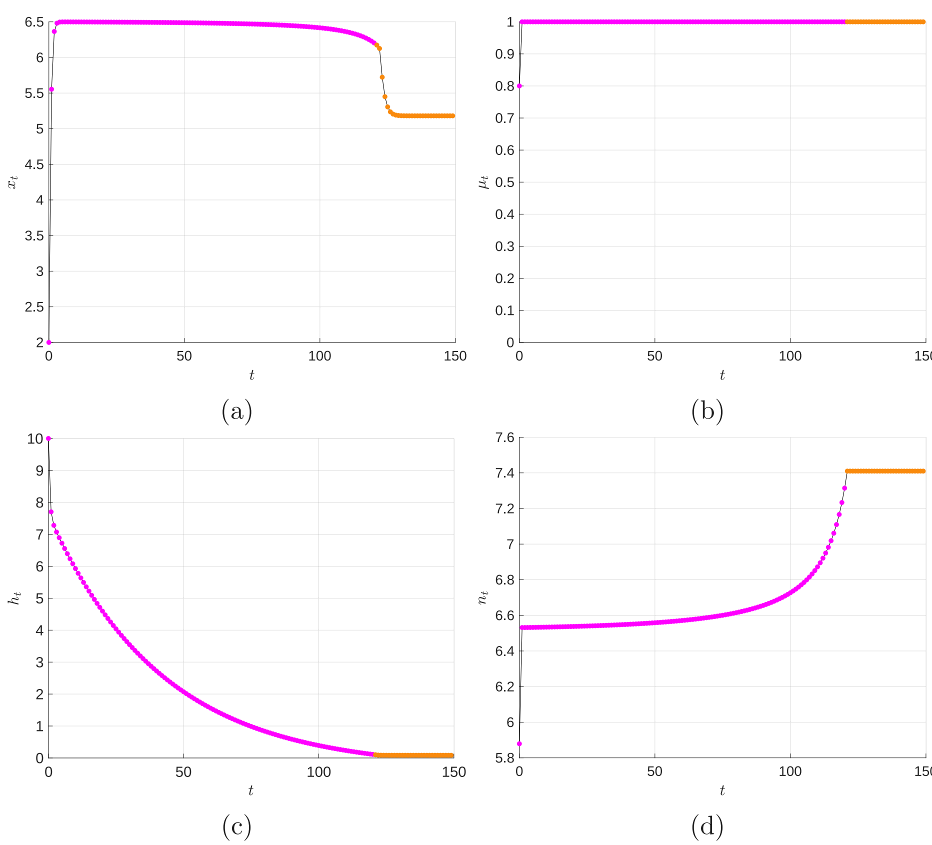

Fig. 7. Dynamics in cases B and D for parameter values $A = 100$, $\alpha = 1/3$, $\beta = 0.1$, $\delta = 0.6$, $\epsilon = 0.1$, $\phi = 0.9$, $\gamma = 0.4$, $\lambda = 0.5$, $\rho = 0.2994$, $\pi = 0.9$, $\theta = 0.3$, and $z = 0.075$ and initial conditions $x_0 = 2$, $\mu_0 = 0.8$, and $h_0 = 10$.

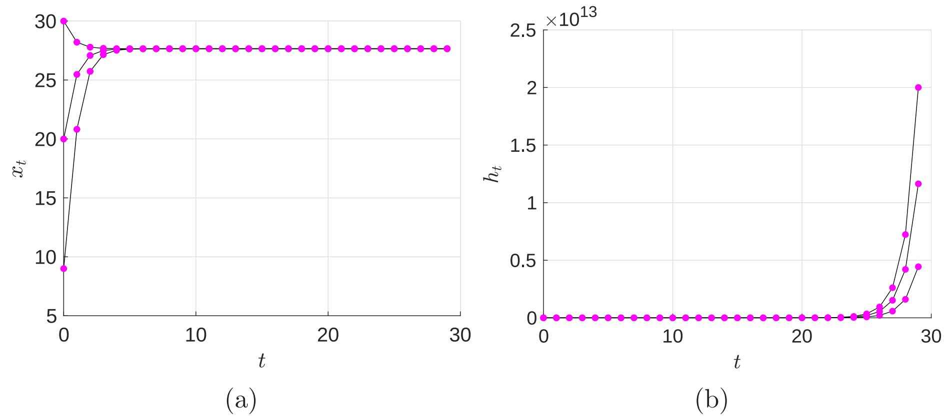

Fig. 8. Several dynamics in case B for parameter values $A = 100$, $\alpha = 0.5$, $\beta = 0.5$, $\delta = 0.5$, $\epsilon = 1$, $\phi = 0.5$, $\gamma = 0.3$, $\lambda = 0.8$, $\rho = 0.8$, $\pi = 0.8$, $\theta = 0.5$, and $z = 0.2$ and different initial conditions.

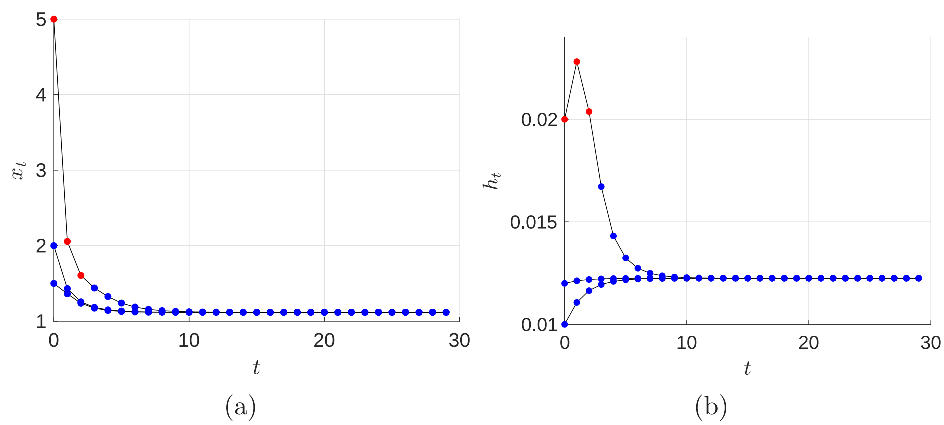

Fig. 9. Several dynamics in case C for parameter values $A = 100$, $\alpha = 0.5$, $\beta = 0.4$, $\delta = 0.5$, $\epsilon = 0.1$, $\phi = 0.6$, $\gamma = 0.7$, $\lambda = 0.99$, $\rho = 0.05$, $\pi = 0.35$, $\theta = 0.2$, and $z = 0.7$ and different initial conditions.

as for the following initial points: $(x_0, \mu_0, h_0) = (1.45, 0.48, 238)$, and $(x_0, \mu_0, h_0) = (7.45, 0.68, 438)$.

As can be clearly seen, the values of x_t converge to the value of $x_B = 2.336527$, and the values of μ_t also converge to 1. The latter result follows from the fact that region B is characterized by $\mu_t = 1$, except for μ_0. However, the dynamics of h_t may diverge. This result is not surprising, as x_t has been defined as the ratio of two quantities that can diverge at the same growth rate, and thus their ratio may converge. Since it is sufficient for just one dynamic to diverge, we can conclude that the equilibrium point E_B for this combination of parameters is a repeller.

To demonstrate the validity of this claim, we repeat the experiment starting from the point $(x_0, \mu_0, h_0) = \left(x_B, \mu_B, h_B + 10^{-6}\right)$. The results are illustrated in Fig. 11, where it is clearly seen that all the dynamics remain constant except for that of h_t.

As explained in Devaney (2022), if E_B were an attractor, the dynamics of h_t would also converge to h_B after several iterations. Instead, we clearly observe that the dynamics of h_t, although only slightly, begin to diverge from this value as the number of iterations increases, confirming that E_B is a repeller.

In conclusion, we can confirm that there exists a combination of parameters for which E_B exists. Regarding the combination found, we observed that E_B is an unstable equilibrium point, as x_t and μ_t converge while h_t can diverge.

This divergence is consistent with the fact that E_B lies within region B, a rich region characterized by the growth of h_t. Although these experiments do not rule out all possible cases, they do support the results presented in Proposition 3.1.

3.6. Sensitivity analysis

This section conducts a sensitivity analysis to assess the robustness of our model. To do so, we propose various bifurcation diagrams, displaying the steady-state solution as a function of a varying model parameter. Considering the large number of parameters involved, we limit our analysis to varying γ between 0 and 1, and π between 10 and 20.

We present the same parameter configuration and initial point in Fig. 12 as in Fig. 5, but with a varying γ between 0 and 1. The results reveal that as γ increases, a transition occurs from region A to C at a steady state for x_t and μ_t. In both cases, the steady-state values of the variables decrease as γ increases.

Through various experiments on different parameter configurations, we find that varying γ between 0 and 1, elicits two types of behavior; that described in Fig. 12, where the steady-state values of x_t and μ_t decrease and that shown in Fig. 13.

In this case, the steady-state values of x_t and μ_t remain constant as γ varies. For example, Fig. 13 shows the same parameter configuration as in Fig. 7, revealing a transition from region B to region D.

Since the parameter configuration in this case satisfies conditions (34) and (35), an asymptotically stable equilibrium point exists in region D. At a steady state, the system converges to region D, wherein the coordinates of the equilibrium point do not depend on γ, explaining the consistent behavior observed in Fig. 13.

Finally, Fig. 14 shows the bifurcation diagrams for the same parameter configuration and initial conditions as in Fig. 2, but with π varying from 10 to 20.

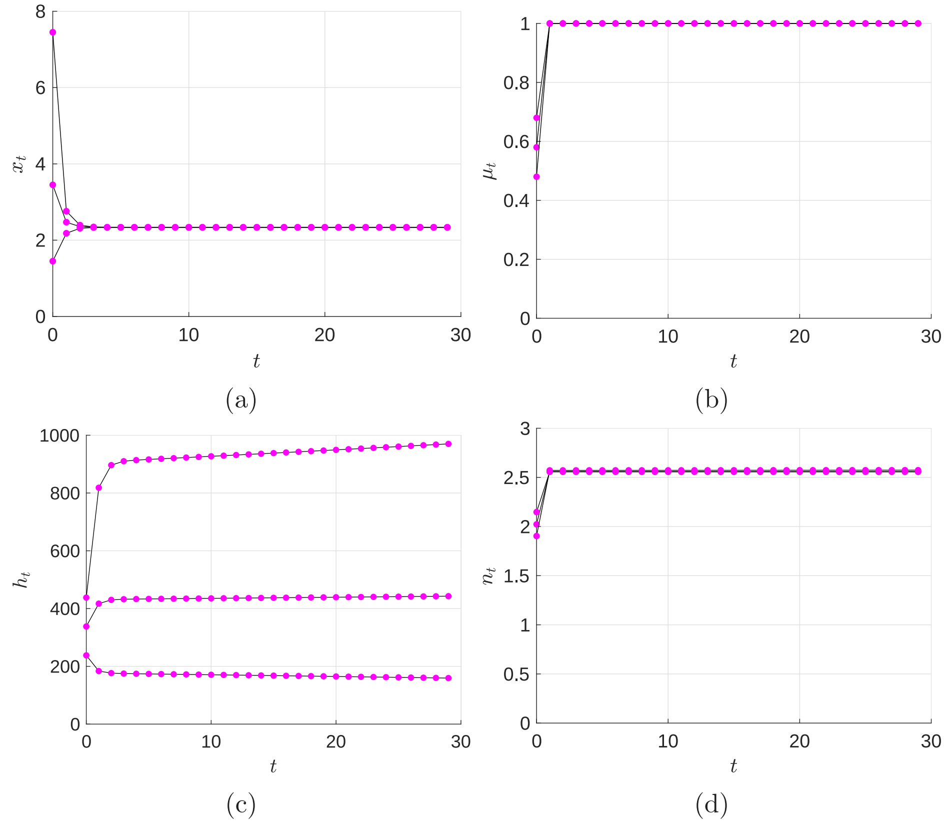

Fig. 10. Dynamics in case B for the proposed combination of parameters and initial values.

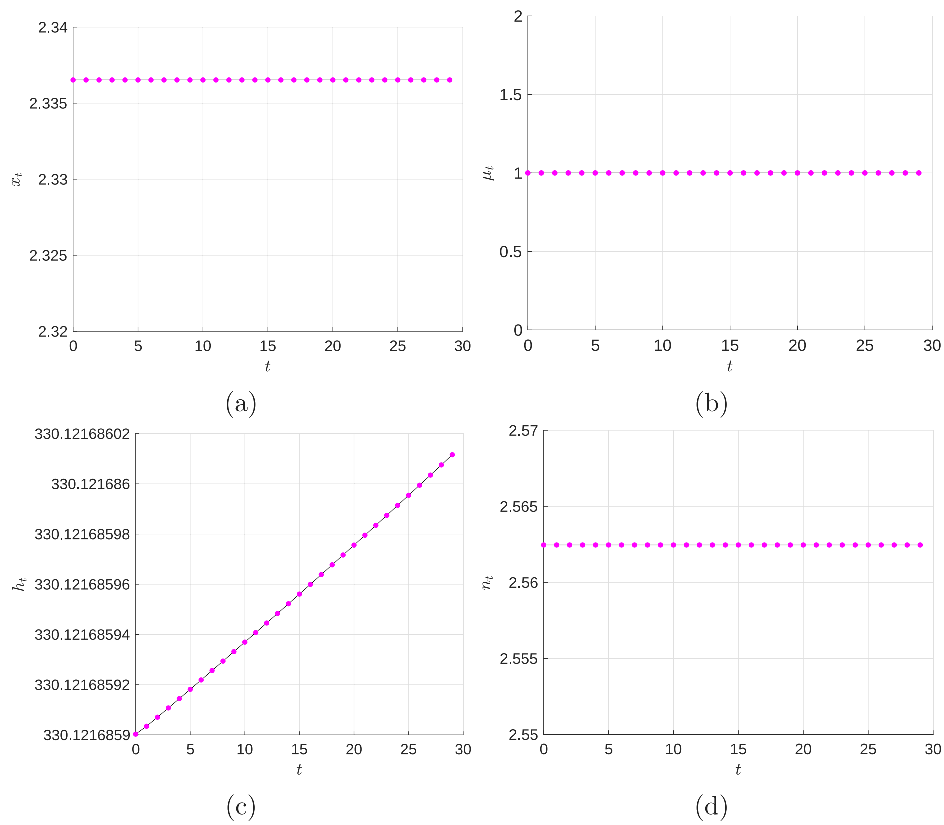

Fig. 11. Dynamics in case B for the proposed combination of parameters and initial values $(x_0, \mu_0, h_0) = (x_B, \mu_B, h_B + 10^{-6})$.

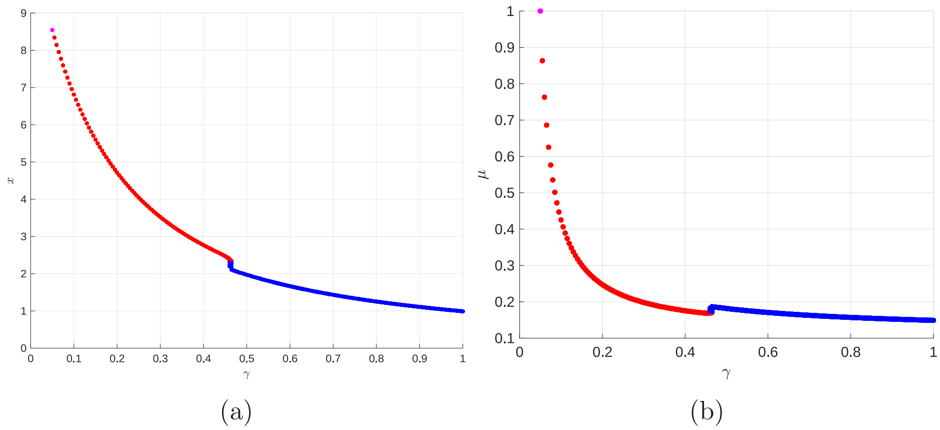

Fig. 12. Bifurcation diagrams for parameter values $A = 100$, $\alpha = 0.25$, $\beta = 0.5$, $\delta = 0.5$, $\epsilon = 0.1$, $\phi = 0.6$, $\lambda = 0.9$, $\rho = 0.1$, $\pi = 0.3$, $\theta = 0.1$, and $z = 0.5$, initial conditions $x_0 = 2$, $\mu_0 = 0.8$, and $h_0 = 2$ and γ varying between 0 and 1.

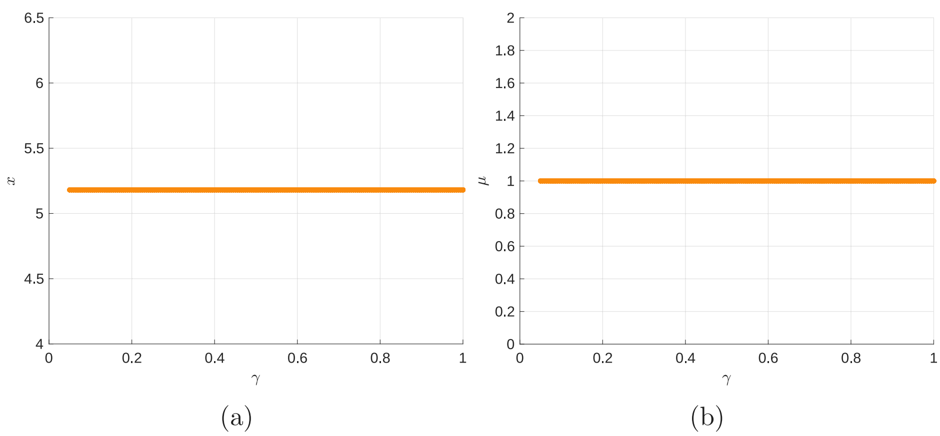

Fig. 13. Bifurcation diagrams for parameter values $A = 100$, $\alpha = 1/3$, $\beta = 0.1$, $\delta = 0.6$, $\epsilon = 0.1$, $\phi = 0.9$, $\lambda = 0.5$, $\rho = 0.2994$, $\pi = 0.9$, $\theta = 0.3$, and $z = 0.075$, initial conditions $x_0 = 2$, $\mu_0 = 0.8$, and $h_0 = 10$ and γ varying between 0 and 1.

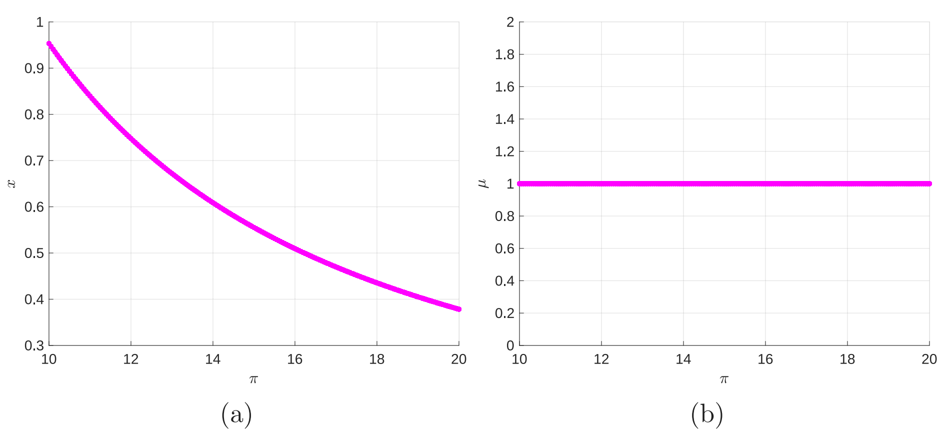

Fig. 14. Bifurcation diagrams for parameter values $A = 100$, $\alpha = 0.5$, $\beta = 0.5$, $\delta = 0.5$, $\epsilon = 1$, $\phi = 0.5$, $\gamma = 0.3$, $\lambda = 0.8$, $\rho = 0.8$, $\theta = 0.5$, and $z = 0.2$, initial conditions $x_0 = 9$, $\mu_0 = 0.8$, and $h_0 = 1$ and π varying between 10 and 20.

As shown, the behavior of μ is constant because we are in region B at a steady state, which is characterized by $\mu_{t+1} = 1$. Conversely, we observe a decreasing trend for x_t as π increases.

We also find this asymptotic behavior for other parameter combinations, leading us to believe that this behavior can be explained by the fact that π values that are not too close to zero cause h_t to diverge. This behavior is clearly observed in the richer regions (A and B). Presumably, for the same reason, π values also satisfy both constraints that characterize region B.

4. Conclusion

Increased longevity has extended the time that grandparents can contribute to raising their grandchildren. In fact, retired parents may invest time in providing childcare. These intergenerational time transfers can play an important role in family decisions. Although limited attention has been given to the role of grandparents as childcare providers, many families rely on them for support in raising their children.

This study investigates how grandparents' childcare efforts influence fertility decisions, elderly individuals' labor supply and ultimately economic development. Our model enables us to consider these variables, wherein adults must decide how many children to raise and how much to invest in their education, while also determining how much time to dedicate to work and caring for their grandchildren in old age. Therefore, we extend the work of Fanti and Gori (2014) by including an endogenous time allocation, focusing on how the elderly choose to divide their time between work and childcare.

Results reveal that increased effective wages in high-income economies tend to raise educational investment in children's human capital, which elevates human capital. Individuals' choice to work in old age does not depend on the state of economic development, as the decision to work or dedicate all one's time to raising grandchildren depends on the relative wage dynamics in high- and low-income countries. In fact, we demonstrate that adults decide to work in old age in both region A (high-income countries) and region C (low-income countries). The decision whether or not to work when elderly depends on comparing the marginal benefit of raising grandchildren (related to the utility it provides) with the opportunity cost of dedicating time to grandchildren (*i.e.* the effective wage they would obtain if they worked when elderly). Therefore, if the growth rate of the present value of the effective wage rate is high, *i.e.* the utility weight of childcare during old age is low, adults will decide to work in their later years.

Following Gori and Michetti (2016) and Blackburn and Cipriani (2002), further research could consider endogenous longevity or introduce inherited tastes.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author Mauro Maria Baldi used ChatGPT in order to proofread the article. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Appendix A

Proof. We begin by expressing an equivalent model, substituting the value of h_{t+1} given by (7) into the objective function. Next, we evaluate s_t in (6) and substitute it into (5). We must also account for the implicit constraint $e_t \geq 0$. After incorporating all these transformations, our model becomes the following:

$$\max \; U_t(c_{1,t}, c_{2,t+1}, \mu_{t+1}, n_t, e_t) = \ln c_{1,t} + \rho \ln c_{2,t+1} + \rho\theta \ln \mu_{t+1} + \beta \ln \tau + \beta\delta \ln(\epsilon + \phi e_t) + \beta(1-\delta)\ln h_t + \lambda \ln n_t \tag{A.1}$$

$$\text{s.t.:} \quad c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} - \frac{\gamma}{r_{t+1}} w_{t+1} h_t \left(1 - \mu_{t+1}\right) + n_t(e_t + z w_t h_t) = w_t h_t (1+\mu_t) \tag{A.2}$$

$$e_t \geq 0 \tag{A.3}$$

$$0 \leq \mu_{t+1} \leq 1. \tag{A.4}$$

We approach the revised model using the Lagrangian multipliers method. To do so, we rephrase constraint $e_t \geq 0$ as $-e_t \leq 0$ to ensure that the Lagrange multipliers associated with inequality constraints are non-negative. The rank of the matrix linked to equality and (potentially) binding inequality constraints is at its maximum. Therefore, qualification constraints are satisfied. Let η_t, χ_t, and ξ_t represent the Lagrange multipliers associated with constraints (A.2)–(A.4). The Lagrangian function $\mathcal{L}_t$ is as follows:

$$\mathcal{L}_t(c_{1,t}, c_{2,t+1}, \mu_{t+1}, e_t, n_t, \eta_t, \chi_t, \xi_t) = U_t(c_{1,t}, c_{2,t+1}, \mu_{t+1}, n_t, e_t) + \\ - \eta_t \left[c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} - \frac{\gamma}{r_{t+1}} w_{t+1} h_t \left(1-\mu_{t+1}\right) + n_t(e_t + z w_t h_t) - w_t h_t (1+\mu_t)\right] + \\ + \chi_t e_t - \xi_t \left(\mu_{t+1} - 1\right). \tag{A.5}$$

Differentiating $\mathcal{L}_t$ with respect to the decision variables $c_{1,t}$, $c_{2,t+1}$, μ_{t+1}, e_t, and n_t and setting these derivatives to zero, we get the following set of equations:

$$\frac{\partial \mathcal{L}_t}{\partial c_{1,t}} = 0 \rightarrow \frac{1}{c_{1,t}} - \eta_t = 0. \tag{A.6}$$

$$\frac{\partial \mathcal{L}_t}{\partial c_{2,t+1}} = 0 \rightarrow \frac{\rho}{c_{2,t+1}} - \frac{\eta_t}{r_{t+1}} = 0. \tag{A.7}$$

$$\frac{\partial \mathcal{L}_t}{\partial \mu_{t+1}} = 0 \rightarrow \frac{\rho\theta}{\mu_{t+1}} - \xi_t - \eta_t \frac{\gamma}{r_{t+1}} w_{t+1} h_t = 0. \tag{A.8}$$

$$\frac{\partial \mathcal{L}_t}{\partial e_t} = 0 \rightarrow \frac{\beta\delta\phi}{\epsilon + \phi e_t} + \xi_t - \eta_t n_t = 0. \tag{A.9}$$

$$\frac{\partial \mathcal{L}_t}{\partial n_t} = 0 \rightarrow \frac{\lambda}{n_t} - \eta_t(e_t + z w_t h_t) = 0. \tag{A.10}$$

Setting the partial derivative with respect to η_t to zero results in the original constraint (A.2).

The complementary slackness conditions associated with the inequality constraints result in the following:

$$\chi_t e_t = 0 \tag{A.11}$$

and

$$\xi_t \left(\mu_{t+1} - 1\right) = 0. \tag{A.12}$$

Finally, non-negativity of the Lagrange multipliers requires the following:

$$\chi_t \geq 0 \tag{A.13}$$

and

$$\xi_t \geq 0. \tag{A.14}$$

The complementary slackness conditions (A.11)–(A.12) serve as the starting point for solving the system of first-order conditions (A.6)–(A.14). According to these conditions, we consider four regions (or cases), denoted as A, B, C and D.

$$R_A : e_t > 0,\ \mu_{t+1} < 1 \rightarrow \chi_t = 0,\ \xi_t = 0$$
$$R_B : e_t > 0,\ \xi_t > 0 \rightarrow \chi_t = 0,\ \mu_{t+1} = 1$$
$$R_C : \chi_t > 0,\ \mu_{t+1} < 1 \rightarrow e_t = 0,\ \xi_t = 0$$
$$R_D : \chi_t > 0,\ \xi_t > 0 \rightarrow e_t = 0,\ \mu_{t+1} = 1$$

We easily obtain the first-order conditions listed next after performing some algebra.

Region A

$$\lambda \neq \beta\delta, \quad \frac{1}{\lambda - \beta\delta}(\beta\delta\phi z w_t h_t - \lambda\epsilon) > 0, \quad (1+\mu_t)\frac{w_t}{w_{t+1}} r_{t+1} < \frac{1+\rho+\lambda}{\rho\theta}\gamma,$$

$$c_{1,t} = \frac{\left[(1+\mu_t)w_t + \gamma \frac{w_{t+1}}{r_{t+1}}\right] h_t}{1 + (1+\theta)\rho + \lambda}, \quad c_{2,t+1} = \frac{\rho\left[(1+\mu_t) r_{t+1} w_t + \gamma w_{t+1}\right] h_t}{1 + (1+\theta)\rho + \lambda},$$

$$\mu_{t+1} = \frac{\rho\theta}{\gamma} \frac{(1+\mu_t)w_t + \gamma \frac{w_{t+1}}{r_{t+1}}}{1+(1+\theta)\rho+\lambda} \frac{r_{t+1}}{w_{t+1}}, \quad e_t = \frac{\beta\delta}{\lambda-\beta\delta} z w_t h_t - \frac{\lambda}{\lambda-\beta\delta}\frac{\epsilon}{\phi}, \text{ and}$$

$$n_t = \frac{\lambda-\beta\delta}{1+(1+\theta)\rho+\lambda} \frac{\left[(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}\right] h_t}{z w_t h_t - \frac{\epsilon}{\phi}}$$

Region B

$$\lambda \neq \beta\delta, \quad \frac{1}{\lambda-\beta\delta}(\beta\delta\phi z w_t h_t - \lambda\epsilon) > 0, \quad (1+\mu_t)\frac{w_t}{w_{t+1}} r_{t+1} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma,$$

$$c_{1,t} = \frac{(1+\mu_t)w_t h_t}{1+\rho+\lambda}, \quad c_{2,t+1} = \frac{\rho(1+\mu_t) r_{t+1} w_t h_t}{1+\rho+\lambda}, \quad \mu_{t+1} = 1,$$

$$e_t = \frac{\beta\delta}{\lambda-\beta\delta} z w_t h_t - \frac{\lambda}{\lambda-\beta\delta}\frac{\epsilon}{\phi}, \text{ and } \quad n_t = \frac{\lambda-\beta\delta}{1+\rho+\lambda}\frac{(1+\mu_t)w_t h_t}{z w_t h_t - \frac{\epsilon}{\phi}}.$$

Region C

$$\beta\delta\phi z w_t h_t - \lambda\epsilon \leq 0, \quad (1+\mu_t)\frac{w_t}{w_{t+1}} r_{t+1} < \frac{1+\rho+\lambda}{\rho\theta}\gamma,$$

$$c_{1,t} = \frac{\left[(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}\right] h_t}{1+(1+\theta)\rho+\lambda},$$

$$c_{2,t+1} = \frac{\rho\left[(1+\mu_t) r_{t+1} w_t + \gamma w_{t+1}\right] h_t}{1+(1+\theta)\rho+\lambda}, \quad \mu_{t+1} = \frac{\rho\theta}{\gamma}\frac{(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}}{1+(1+\theta)\rho+\lambda}\frac{r_{t+1}}{w_{t+1}},$$

$$e_t = 0, \text{ and } \quad n_t = \frac{\lambda}{1+(1+\theta)\rho+\lambda}\left(1+\mu_t + \frac{\gamma}{r_{t+1}}\frac{w_{t+1}}{w_t}\right)\frac{1}{z}.$$

Region D

$$\beta\delta\phi z w_t h_t - \lambda\epsilon \leq 0, \quad (1+\mu_t)\frac{w_t}{w_{t+1}} r_{t+1} \geq \frac{1+\rho+\lambda}{\rho\theta}\gamma, \quad c_{1,t} = \frac{(1+\mu_t)w_t h_t}{1+\rho+\lambda},$$

$$c_{2,t+1} = \frac{\rho(1+\mu_t) r_{t+1} w_t h_t}{1+\rho+\lambda}, \quad \mu_{t+1} = 1, \quad e_t = 0, \text{ and}$$

$$n_t = \frac{\lambda}{1+\rho+\lambda}(1+\mu_t)\frac{1}{z}.$$

To compute second-order conditions, we first determine the Hessian matrix H of the Lagrangian function, given by:

$$H = \begin{bmatrix} -\frac{1}{c_{1,t}^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\rho}{c_{2,t+1}^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\rho\theta}{\mu_{t+1}^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\beta\delta\phi^2}{(\epsilon+\phi e_t)^2} & -\eta_t \\ 0 & 0 & 0 & -\eta_t & -\frac{\lambda}{n_t^2} \end{bmatrix}$$

For the stationary points to be local maxima, the Hessian matrix must be negative definite on the set defined by equality and active inequality constraints. We find that the signs of the first four leading principal minors alternate as $(-1)^k$, where k is the order of the minor. Therefore, it is sufficient to ensure that the determinant is negative. This implies satisfying the following inequality:

$$\frac{\beta\delta\phi^2}{(\epsilon+\phi e_t)^2}\frac{\lambda}{n_t^2} - \eta_t^2 > 0. \tag{A.15}$$

Substituting the values of e_t, n_t and η_t for case A into (A.15), we obtain the following inequality after performing some algebra:

$$\frac{[1+(1+\theta)\rho+\lambda]^2}{\left[(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}\right]^2 h_t^2}\left(\frac{\lambda}{\beta\delta} - 1\right) > 0,$$

which holds true if $\lambda - \beta\delta > 0$. Employing a similar approach in case B yields a distinct inequality, which is also satisfied under the condition $\lambda - \beta\delta > 0$.

Substituting the values of e_t, n_t and η_t for case C into (A.15), we obtain the following inequality after performing some algebra:

$$\frac{[1+(1+\theta)\rho+\lambda]^2}{\left[(1+\mu_t)w_t + \gamma\frac{w_{t+1}}{r_{t+1}}\right]^2}\left[\frac{\beta\delta}{\lambda}\left(\frac{\phi}{\epsilon}\right)^2 z^2 w_t^2 - \frac{1}{h_t^2}\right] > 0.$$

This inequality is satisfied if the following holds:

$$w_t h_t > \frac{\epsilon}{\phi z}\sqrt{\frac{\lambda}{\beta\delta}}. \tag{A.16}$$

Employing the same approach in case D produces a different inequality but results in the same second-order condition as in (A.16). By combining first- and second-order conditions, the claim is established. In particular, for each region, the value of s_t can be obtained using (6). □

Appendix B

Proof. Increasing time t by one unit in (14), we obtain the following:

$$m_{t+1} = h_{t+1} + \frac{\gamma(1-\mu_{t+1})h_t}{n_t}. \tag{B.1}$$

Specifically, when $\mu_{t+1} = 1$ (cases B and D), (B.1) reduces to $m_{t+1} = h_{t+1}$.

By substituting (17) and (18) into (B.1), we can express m_{t+1} in terms of the ratios k_t/m_t and k_{t+1}/m_{t+1}.

Similarly, by substituting (17) and (18) into the formula for μ_{t+1}, we can express μ_{t+1} in terms of ratios k_t/m_t and k_{t+1}/m_{t+1}. In cases B and D, this relationship is limited to $\mu_{t+1} = 1$.

Performing the same substitution in (3) and considering the assumption $\bar{h}_t = h_t$, we can express μ_{t+1} in terms of the ratios k_t/m_t. In cases C and D, this relationship is limited to $h_{t+1} = \pi\epsilon^\delta h_t^{1-\delta}$.

Finally, by substituting the values of s_t and n_t from the first-order conditions into (19) for each case, and then substituting the values of w_t and r_t from (17) and (18), respectively, we can express k_{t+1} in terms of ratios k_t/m_t and k_{t+1}/m_{t+1}. □

Appendix C

Proof. We only prove the theorem for case C, and similar steps apply for the remaining regions. Dividing (22a) by (22b) on both sides, we obtain the following:

$$\frac{k_{t+1}}{m_{t+1}} = \frac{\frac{z}{\lambda}\frac{(1+\theta)\rho(1+\mu_t)A\alpha\frac{k_t^\alpha}{m_t^\alpha} - \gamma(1+\lambda)\frac{k_{t+1}}{m_{t+1}}}{(1+\mu_t)A\alpha\frac{k_t^\alpha}{h_t^\alpha} + \gamma\frac{k_{t+1}}{m_{t+1}}}A(1-\alpha)\frac{k_t^\alpha}{m_t^\alpha}}{\pi\epsilon^\delta h_t^{-\delta} + \frac{z}{\lambda}\frac{A\alpha\frac{k_t^\alpha}{m_t^\alpha}}{\frac{k_{t+1}}{m_{t+1}}}\frac{\gamma(1+\rho+\lambda)\frac{k_{t+1}}{m_{t+1}} - \rho\theta(1+\mu_t)A\alpha\frac{k_t^\alpha}{m_t^\alpha}}{(1+\mu_t)A\alpha\frac{k_t^\alpha}{m_t^\alpha} + \gamma\frac{k_{t+1}}{m_{t+1}}}}. \tag{C.1}$$

By defining x_t as the ratio k_t/m_t, (C.1) can be reformulated as follows:

$$x_{t+1} = \frac{\frac{z}{\lambda}\frac{(1+\theta)\rho(1+\mu_t)A\alpha x_t^\alpha - \gamma(1+\lambda)x_{t+1}}{(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}}A(1-\alpha)x_t^\alpha}{\pi\epsilon^\delta h_t^{-\delta} + \frac{z}{\lambda}\frac{A\alpha x_t^\alpha}{x_{t+1}}\frac{\gamma(1+\rho+\lambda)x_{t+1} - \rho\theta(1+\mu_t)A\alpha x_t^\alpha}{(1+\mu_t)A\alpha x_t^\alpha + \gamma x_{t+1}}}$$

Extending the same substitution to the remaining equations yields the desired result. □

Appendix D

Proof. We denote, by x_B, μ_B, and h_B the coordinates of equilibrium points satisfying the condition $\mathbf{x} = f(\mathbf{x})$ in this region, that is, those points satisfying the following system:

$$\begin{cases} x_B = \dfrac{\rho}{\pi} \dfrac{\left[zA(1-\alpha)x_B^\alpha h_B - \frac{\epsilon}{\phi}\right]^{1-\delta}}{(\beta\delta\phi)^\delta(\lambda-\beta\delta)^{1-\delta}h_B^{1-\delta}} & \text{(D.1a)} \\ \mu_B = 1 & \text{(D.1b)} \\ h_B = \pi\left(\dfrac{\beta\delta\phi}{\lambda-\beta\delta}\right)^\delta \left[zA(1-\alpha)x_B^\alpha h_B - \dfrac{\epsilon}{\phi}\right]^\delta h_B^{1-\delta} & \text{(D.1c)} \end{cases}$$

For convenience, we define F as follows:

$$F = zA(1-\alpha)x_B^\alpha h_B - \frac{\epsilon}{\phi}.$$

$h_B = 0$ is a solution to (D.1c), but it is not acceptable as it would make the denominator of (D.1a) zero. Therefore, we can divide both sides of (D.1c) by h_B and, after some steps, obtain the following:

$$h_B = \pi^{\frac{1}{\delta}} \frac{\beta\delta\phi}{\lambda-\beta\delta} F. \tag{D.2}$$

This implies the following:

$$F = \frac{1}{\pi^{\frac{1}{\delta}}} \frac{\lambda-\beta\delta}{\beta\delta\phi} h_B. \tag{D.3}$$

We can rewrite (D.1a) as:

$$x_B = \frac{\rho}{\pi} \frac{F^{1-\delta}}{(\beta\delta\phi)^\delta(\lambda-\beta\delta)^{1-\delta}h_B^{1-\delta}}. \tag{D.4}$$

Plugging (D.3) into (D.4), after some steps we obtain the following:

$$x_B = \frac{\rho}{\beta\delta\phi} \frac{1}{\pi^{\frac{1}{\delta}}}.$$

Substituting this value into (D.2), after some steps, we obtain the following:

$$h_B = \frac{\epsilon}{\phi} \frac{1}{zA(1-\alpha)\left(\frac{\rho}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}\right)^\alpha - \frac{\lambda-\beta\delta}{\beta\delta\phi}\frac{1}{\pi^{\frac{1}{\delta}}}}.$$

Finally, we must ensure that the point E_B that was just determined actually exists. To do so, we simply substitute the E_B coordinates into the inequalities describing region B. We obtain (32) and (33) following some straightforward steps.

Concerning case D, we denote, by x_D, μ_D, and h_D the coordinates of equilibrium points satisfying the condition $\mathbf{x} = f(\mathbf{x})$ in this region; that is, those points that satisfy the following system:

$$\begin{cases} x_D = \dfrac{\rho}{\pi} \dfrac{\left[zA(1-\alpha)x_D^\alpha h_D - \frac{\epsilon}{\phi}\right]^{1-\delta}}{(\beta\delta\phi)^\delta(\lambda-\beta\delta)^{1-\delta}h_D^{1-\delta}} & \text{(D.5a)} \\ \mu_D = 1 & \text{(D.5b)} \\ h_D = \pi\left(\dfrac{\beta\delta\phi}{\lambda-\beta\delta}\right)^\delta \left[zA(1-\alpha)x_D^\alpha h_D - \dfrac{\epsilon}{\phi}\right]^\delta h_D^{1-\delta} & \text{(D.5c)} \end{cases}$$

While Eq. (D.5c) is satisfied for $h_D = 0$, this solution is not acceptable as it would make the (D.5a) denominator equal zero. Therefore, since $h \neq 0$, we divide both sides of (D.5c) by h_D and after some straightforward steps, we obtain the following:

$$h_D = \epsilon\pi^{\frac{1}{\delta}}.$$

Substituting the value of h_D into (D.5a), we obtain the following after some straightforward steps.

$$x_D = \left[\frac{\rho}{\lambda} zA(1-\alpha)\right]^{\frac{1}{1-\alpha}}$$

Since $\mu_{t+1} = 1$ in the original system, this equation would produce a row of zeros in the Jacobian matrix. Therefore, we construct the Jacobian matrix associated with the system in case D by considering only Eqs. (27a) and (27c). Substituting the coordinates of E_D into this Jacobian matrix, we obtain the following:

$$J_D(x_D, h_D) = \begin{bmatrix} \alpha & \dfrac{\delta\left[\frac{\rho}{\lambda}zA(1-\alpha)\right]^{\frac{1}{1-\alpha}}}{\epsilon\pi^{\frac{1}{\delta}}} \\ 0 & 1-\delta \end{bmatrix}$$

Therefore, the equilibrium point E_D is asymptotically stable; however, it is also necessary to ensure that this point exists. To do so, we substitute its coordinates into the conditions governing region D. Following some simple steps, we obtain the conditions (34) and (35). □

Data availability

I have shared the link to my data/code at the Attach File step.

Elderly labor supply, endogenous grandparental childcare, and fertility in an OLG model. (Original data) (Mendeley Data)

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