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The role of grandparents in grandchildren's education for human capital accumulation in an overlapping generations model

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ABSTRACT

We develop an overlapping generations model to explore the role of grandparents in grandchildren's education and its impact on human capital growth. We examine the quantity–quality (Q–Q) trade-off faced by parents in choosing the number and education of children, incorporating an active role for grandparents. Findings underscore the significance of the elderly in human capital accumulation, fertility, and economic growth. When grandparents invest more time, resources are freed, fostering greater human capital growth and mitigating the effects of the Q–Q trade-off.

1. Introduction

The intuitive concept of a trade-off between quantity and quality of children is a central link in human fertility theories. In economic literature, the notion has been attributed to the economist Gary Becker (Becker, 1960). Prior to Becker, fertility was a widely studied topic outside of economic analysis, partly because data showing simultaneously declining fertility rates and progressively rising income levels seemed to run counter to the assumption of economic rationality. In fact, one of the most striking aspects of the analysis of demographic transition is to observe that declining fertility seems to coexist with an increase in the resources that parents devote to the raising and education of each of their children. Becker explained these trends by formalizing the idea that parents can derive utility from both the quantity and quality of their children, thus succeeding in explaining lower fertility, higher income, and higher spending per child. From Becker's pioneering papers (Becker, 1960; Becker and Lewis, 1973) onward, the quantity–quality (Q–Q) trade-off of children has been a central theme in all fertility studies of both high-income and low-income countries. Indeed, economic analysis has progressively developed the idea that parents face a trade-off between quantity and quality of offspring, a trade-off whose balance shifts from the former to the latter as an economy moves toward more developed stages of its development process. Numerous analyses, which support the Q–Q trade-off have been conducted over the years, such as Rosenzweig and Wolpin (1980), Hanushek (1992), Bleakley and Lange (2009), and Becker et al. (2010).

As Doepke et al. (2023) highlight "new empirical regularities have emerged that cannot be accounted for by baseline economic theories of fertility". The "first" generation of economic models of fertility show a negative relationship between income levels and fertility but today's data show that the income-fertility relationship is now largely flat in many countries and growing in cross-section studies of high-income countries. "The new facts about fertility behavior in high-income countries do not mean that the ideas of a quantity–quality trade-off or of a central role of the opportunity cost of mothers' time were wrong. The trade-offs emphasized by these models still exist and continue to be important in explaining fertility behavior in many places, including lower-income countries. What has changed, however, is that these trade-offs no longer drive the major variation in the data for high-income countries". The authors highlight several factors that can help mothers combine a career with a larger family, i.e. "the availability of public child care and other supportive family policies; greater contributions from fathers in providing childcare; social norms in favor of working mothers; and flexible labor markets". Our work fits into this recent strand of research by proposing an important additional element, i.e., the role of grandparents in the education of grandchildren, which may explain the attenuation of this trade-off since the presence of grandparents replacing parents in their human capital formation function frees up resources that can be directed toward making more children.

Traditionally, research on the intergenerational transmission of advantages and disadvantages has primarily focused on transfers from parents to children. Generally, greater parental resources enable them to offer emotional, educational, financial, material, and social support to their children, fostering health (Kahn et al., 2005), social-emotional well-being (Mistry et al., 2002), and cognitive development.

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Recently, an increasing body of research highlights the limitations of the traditional two-generation framework, which confines investigations to the parent–child relationship. Instead, these studies propose an alternative framework that examines the influence of grandparents and even ancestors beyond parents in shaping educational and social stratification processes (Mare, 2011; Pfeffer, 2014). This shift is particularly significant given the rising trend of longevity, or longer life spans, which can profoundly impact the roles and dynamics of grandparents within families.

With longer life expectancies, grandparents have more opportunities to form extended relationships with their grandchildren. This extended time fosters deeper connections and allows for the sharing of a broader range of life experiences. Additionally, grandparents can act as mentors and guides, imparting wisdom, advice, and life lessons gleaned over the years, thereby enriching the personal and intellectual development of their grandchildren. Longer life expectancies also enable grandparents to serve as positive aging role models, challenging aging stereotypes and showcasing the potential for ongoing growth, learning, and contributions to both family and society. Such a consideration is taking more and more relevance in high-income developed economies due to the increasing quantity and quality of life level of old people.

There exists empirical evidence that "extended" family members, particularly grandparents, play a major role in the economic, cultural, and social conditions of grandchildren (see e.g. Bengtson (2001)). In particular, grandparents' resources have a direct effect on grandchildren's achievements such as cognitive development (Modin and Fritzell, 2009; Ferguson and Ready, 2011) but also performance achieved in schooling (Falbo, 2014). Finally, the presence of a grandparent proximity effect has been found in many of the places where it could be looked for: rural China (Zeng and Xie, 2014), Sweden (Lindahl et al., 2015), Great Britain (Chan and Boliver, 2013), Germany (Hertel and Groh-Samberg, 2014), and Denmark (Boserup et al., 2016).

For these reasons, in this paper, we analyze the role of grandparents in grandchildren's education in the framework of child Q–Q trade-off. More precisely, we consider that not only parents but also grandparents can contribute to children's education. In such a way, we are able to investigate how active aging can be a source of economic growth passing through the increase in the human capital mechanism creation. Inspired by the existing literature showing that knowledge transfers from grandparents can affect children's quality labor force once adult (Ferrie et al., 2021), the main scope of the present work is then to formalize an overlapping generations (OLG) model able to describe how and in which measure increases in active aging can influence the educational process, human capital creation, and, hence, economic growth. In fact, unlike previous contributions, we incorporate the time spent by grandparents on the educational level of grandchildren in the law describing human capital creation. The new channel consists of a transmission of skills, competencies, and abilities from the old generation to the young generation, depending on both the amount of time grandparents spend with their grandchildren and the "quality" of this time, a quality related to the educational level of the grandparents themselves. Such a component plays a crucial role. In fact, the more time grandparents devote to the education of their grandchildren and the more educated they are, the more education is passed on from older to younger generations who will, as a result, have a higher level of human capital that will generate, ceteris paribus, greater economic growth. Therefore, in our economic framework, we aim to elucidate how and to what extent increases in grandparents' involvement in their grandchildren's educational process contribute to human capital accumulation and, consequently, economic growth.

In the resulting dynamic setting, children will become more educated adults and, once they become grandparents, they will be able to transfer a higher education to their grandchildren. The mechanism repeats over time and will produce, in the long term, a virtuous cycle. On the one hand, we show that human capital accumulation plays a fundamental role in economic growth thus confirming results first proposed by Lucas (1988) and we prove that a positive relationship between the time spent by grandparents in educating grandchildren and human capital accumulation level is exhibited. On the other hand, the Q–Q trade-off is confirmed, but the presence of the active role of grandparents in the education of grandchildren allows, ceteris paribus, to "mitigate" the effects of this trade-off.

Following the literature, e.g. Yakita (2010), we consider an economy in which the parental utility associated with rearing children is greater than the utility deriving from children's human capital. In this context, it turns out, as usual, the income level of the country considered is very important. Indeed, in low-income countries, parents tend to focus solely on fostering their children's physical growth rather than investing in their education. In this scenario, when grandparents spend a significant amount of time with children, the number of children remains unchanged, even if they attain a higher level of education, as grandparents play a crucial role in shaping their human capital. This leads to a less stringent (with regard to the quality) Q–Q trade-off.

Similarly, in high-income countries, the education provided by grandparents partially replaces that of parents, allowing the latter to have more economic resources to devote to generate more children. Again, the active role played by grandparents allows the Q–Q trade-off to be less stringent (with regard to the quantity). The use of OLG models allows us to analyze the child quantity–quality trade-off which is essential for policymakers and economists to design effective policies that promote economic growth, social welfare, and intergenerational equity. The interactions between different generations within a population are central to the analysis of OLG models and provide valuable insights into the economic challenges and opportunities faced by societies over time.

The paper proceeds as follows. Section 2 illustrates the setup of the model. Section 3 presents both the analytical and numerical results. Section 4 concludes the paper.

2. The model

We consider a production economy populated by overlapping generations of homogeneous agents who live for three periods: childhood, adulthood, and old age. As usual, we consider that this economy produces a homogeneous physical good that is either consumed or invested to build future capital.[1]

Agents receive education in the first period, work and rear children in the second, and retire in the third. The new element is due to the consideration that during the third period (retirement) the older generation can devote time both to leisure and to educating children. Time is discrete and denoted by $t \in \mathbb{N}$, while the length of each period is normalized to one.

2.1. Individuals

At time t the economy consists of N_{t-1} old agents, N_t adult agents and N_{t+1} young agents. Assuming that adults at time t rear n_t children, the number of adult agents at time $t+1$ evolves according to the following rule:

$$N_{t+1} = n_t N_t, \tag{1}$$

where n_t represents the number of children per adult at time t. We consider unisex individuals, setting aside gender differences and that all agents are alike, except for their ages (i.e., their dates of birth). Adults choose how much time to devote to working and rearing children. Parents are "altruistic", as they care not only about consumption and leisure, but also about their children and the human capital of their children (Becker and Tomes, 1976).

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[1] See e.g. De La Croix and Michel (2002).

Consumption of the old period is financed by returns to the savings accumulated during adulthood (Galor and Weil, 2000). In our model, we simplify the analysis by considering that old agents can live only with the amount of resources saved when young, avoiding pension benefits (for a detailed analysis of the dynamic properties of an OLG economy with endogenous fertility and fertility-related pensions (see e.g Fanti and Gori (2012, 2013)). In addition, we assume that all goods are perishable and that agents can only transfer value across time by means of capital markets. In the second period of life, adults work and divide their income between rearing children and educating them.

The new element of the present work is the role played by grandparents in educating their grandchildren. In fact, in the third period, old agents can devote their time to leisure time or, alternatively, they can spend time educating grandchildren. Hence we assume that each child is raised by parents but he/she can be educated either by parents, through education outside the family, or by grandparents spending part of their time with grandchildren.

To summarize, adults derive utility from the consumption they will enjoy in the adulthood period and in old age, from the number of children and their "quality" (human capital), and from the social interactions enjoyed during the time left educating children in old age, which hereafter we will call "leisure time".

The lifetime utility of an individual of generation t is then represented by the following function:

$$U_t\left(c_{1,t},\, c_{2,t+1},\, n_t,\, e_t\right) = \ln c_{1,t} + \rho\{\ln c_{2,t+1} + \theta \ln(1-\mu)\} + \beta \ln h_{t+1} + \lambda \ln n_t, \tag{2}$$

where $c_{1,t}$ is the consumption during adulthood age, $c_{2,t+1}$ is the consumption during retirement, n_t is the number of children, h_{t+1} is the human capital of the offspring and all variables are not negative. We underline the role of the parameter $\mu \in (0,1)$ representing the fraction of time devoted to educating grandchildren and thus $(1-\mu)$ is the fraction of leisure of old-age agents.

Parameter $\rho \in (0,1)$ represents the discount factor, $\lambda \in (0,1)$ and $\beta \in (0,1)$ respectively represent the weight attached to quantity and quality of children and $\theta \in (0,1)$ is the weight given to the leisure time. It is worth noting that we have highlighted in (2) the dependency of the utility function on the decision variables $c_{1,t}$, $c_{2,t+1}$, n_t, and e_t.

The budget constraint in the second young adulthood period of the individual is:

$$w_t h_t = c_{1,t} + s_t + n_t(e_t + z w_t h_t), \tag{3}$$

where s_t is the agents' life-cycle saving, w_t is the wage rate for labor, so that $w_t h_t$ represents the wage rate for effective labor, $z \in (0,1)$ denotes the child-rearing cost per child, which is assumed, following Hirazawa and Yakita (2017), to be equal to a constant proportion of the wage income. Finally, e_t is the per-child educational expenditure. Given parents' income, the cost of rearing children and the cost of education distracts resources from alternative uses, i.e., present and future consumption. Denoting r_{t+1} the interest factor in the period $t+1$, the budget constraint of the third period is:

$$c_{2,t+1} = r_{t+1} s_t. \tag{4}$$

Following De La Croix and Doepke (2003), Yakita (2010) and Hirazawa and Yakita (2017), we assume that education consists of two parts. The first part is the fruit of the time devoted by grandparents to educating grandchildren. This component derives from the quantity and "quality" (which we approximate by grandparent education, i.e., their human capital h_{t-1}) of time devoted to grandchildren by grandparents. The second component relies on the offspring's learning from educational institutions and thereby entails an expenditure that is subject to budget constraints. In other words, parents delegate the formal education of the offspring to the educational system (De la Croix and Doepke, 2004). The average level of human capital in the economy, $\bar{h}_t$, also plays a role due to spillover effects, i.e., due to the fact that the learning process is more effective if a child interacts with more educated individuals. The human capital of an individual represents the effective labor productivity of working at time $t+1$ and it is produced according to the following function:

$$h_{t+1} = \pi\left(\mu h_{t-1} + \phi e_t\right)^{\delta} \bar{h}_t^{1-\delta}, \tag{5}$$

where h_{t-1} is the stock of human capital of old generation, e_t is the educational expenditure, and $\bar{h}_t$ is the average stock of human capital of generation t representing the spillover effect from society. The term $(\mu h_{t-1} + \phi e_t)$ represents the education provided by the family and the human capital transmission mechanism coming from grandparents whose relevance depends on the fraction μ and on the educational level of grandparents h_{t-1}.[2] Parameter $\pi > 0$ indicates the technology of production of human capital, $\delta \in (0,1)$ indicates the weight of the role of "family" in the education of the children, $(1-\delta)$ is the productivity of the average level of human capital in the economy, and $\phi \in (0,1]$ expresses the importance of resources devoted to education increasing the effectiveness of education.

The problem for the adult generation at time t is to choose how much to consume $c_{1,t}$ and, hence, the saving level s_t or the consumption level when old $c_{2,t+1}$, the number of children n_t and the parental educational expenditure e_t in order to maximize the lifetime utility U_t under budget constraints. It is summarized in Section 2.2.

2.2. The optimization model

In this section, we provide a mathematical formulation of the optimization problem involving the adult generation at time $t \in \mathbb{N}$. Given the large number of parameters and variables involved, we first list them and recall their meaning.

Parameters

- $A > 0$: the total productivity factor.
- $\mu \in (0,1)$: the fraction of time devoted to educating grandchildren.
- $\rho \in (0,1)$: the discount factor.
- $\lambda \in (0,1)$: the weight associated to the quantity of children in the objective function.
- $\beta \in (0,1)$: the weight associated to the quality of children in the objective function.
- $\theta \in (0,1)$: the weight associated to the leisure time in the objective function.
- $z \in (0,1)$: the child-rearing cost per child.
- $\pi > 0$: the technology coefficient of production of human capital.
- $\delta \in (0,1)$: the weight of the role of "family" in the education of children.
- $\phi \in (0,1]$: expresses the importance of resources devoted to education increasing the effectiveness of education.

Decision variables

- $c_{1,t}$: the consumption during adulthood age.
- $c_{2,t+1}$: the consumption during retirement.
- n_t: the number of children.
- e_t: the educational expenditure.

Other variables

- h_{t-1}: the stock of human capital of the old generation.
- h_t: the stock of human capital of the adult generation.
- $\bar{h}_t$: the stock of human capital of the adult generation.
- h_{t+1}: the human capital of the offspring.
- w_t: the wage rate for labor.

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[2] The two inputs are perfect substitutes as proposed in other works of this kind, see e.g. Yakita (2010).

- r_{t+1}: the interest factor in period $t+1$.
- s_t: the agents' life-cycle saving.

Our model consists in maximizing the utility function given by (2) depending on the decision variables $c_{1,t}$, $c_{2,t+1}$, n_t, and e_t. It is a constrained maximization problem subject to the following constraints: a budget constraint given by Eq. (3), Eq. (4) involving the interest factor, and Eq. (5) expressing the evolution of human capital. Putting all these considerations together, we get the following optimization problem:

$$\max \quad U_t(c_{1,t}, c_{2,t+1}, n_t, e_t) = \ln c_{1,t} + \rho \left[\ln c_{2,t+1} + \theta \ln(1-\mu)\right] + \beta \ln h_{t+1} + \lambda \ln n_t \tag{6}$$

$$\text{s. t.:} \quad w_t h_t = c_{1,t} + s_t + n_t(e_t + z w_t h_t) \tag{7}$$

$$c_{2,t+1} = r_{t+1} s_t \tag{8}$$

$$h_{t+1} = \pi \left(h_{t-1}\mu + \phi e_t\right)^{\delta} \bar{h}_t^{1-\delta}. \tag{9}$$

It is worth noting that all the involved variables cannot be negative. Furthermore, since $c_{1,t}$, $c_{2,t+1}$, h_{t+1}, and n_t are the argument of natural logarithms in the objective function, only e_t can be exactly 0.

Notice that from the previous assumption of identical agents within a generation, we have that $\bar{h}_t = h_t$.

The constrained maximization problem gives the following solutions (see Appendix A for a sketch of the proof).

Proposition 2.1. *Let* $\gamma\left(w_t, h_t, h_{t-1}\right) = \frac{\beta\delta}{\lambda} z w_t h_t - \frac{\mu}{\phi} h_{t-1}$.

(i) If $\frac{\beta\delta}{\lambda} \neq 1$, *then the first-order condition for the constrained maximization problem requires:*

$$c_{1,t} = \begin{cases} \frac{1}{1+\rho+\lambda}\, w_t h_t & \textit{if } \gamma \neq 0 \\ \frac{1}{1+\rho+\lambda} \frac{1}{z} \frac{\lambda}{\beta\delta} \frac{\mu}{\phi} h_{t-1} & \textit{if } \gamma = 0, \end{cases} \tag{10}$$

$$c_{2,t+1} = \begin{cases} \frac{\rho}{1+\rho+\lambda}\, r_{t+1} w_t h_t & \textit{if } \gamma \neq 0 \\ \frac{\rho}{1+\rho+\lambda} \frac{1}{z} \frac{\lambda}{\beta\delta} \frac{\mu}{\phi} r_{t+1} h_{t-1} & \textit{if } \gamma = 0, \end{cases} \tag{11}$$

and

$$s_t = \begin{cases} \frac{\rho}{1+\rho+\lambda}\, w_t h_t & \textit{if } \gamma \neq 0 \\ \frac{\rho}{1+\rho+\lambda} \frac{1}{z} \frac{\lambda}{\beta\delta} \frac{\mu}{\phi} h_{t-1} & \textit{if } \gamma = 0. \end{cases} \tag{12}$$

(ii) Let $\frac{\beta\delta}{\lambda} \in (0, 1)$. *Then conditions (i) are also sufficient and*

$$e_t = \begin{cases} \frac{\beta\delta}{\lambda-\beta\delta} z w_t h_t - \frac{\lambda}{\lambda-\beta\delta} \frac{\mu}{\phi} h_{t-1} & \textit{if } \gamma > 0 \\ 0 & \textit{if } \gamma \leq 0, \end{cases} \tag{13}$$

while

$$n_t = \begin{cases} \frac{\lambda-\beta\delta}{1+\rho+\lambda} \frac{1}{z - \frac{\mu}{\phi} \frac{h_{t-1}}{w_t h_t}} & \textit{if } \gamma > 0 \\ \frac{\lambda}{1+\rho+\lambda} \frac{1}{z} & \textit{if } \gamma \leq 0. \end{cases} \tag{14}$$

Notice that, the hypothesis $\frac{\beta\delta}{\lambda} \in (0, 1)$ in Proposition 2.1, part (ii) is the same as in Yakita (2010). This condition may also have an economic interpretation. Recalling that β is the weight in the utility function that parents attribute to their children's human capital and that δ is the productivity of education e_t in the production of human capital h_{t+1}, $\beta\delta$ is the utility associated with the human capital produced through education by parents. So condition $\frac{\beta\delta}{\lambda} \in (0, 1)$ states that the parental utility associated with making children is greater than the utility associated with human capital.

Notice that $\frac{dn_t}{dw_t} < 0$ and $\frac{de_t}{dw_t} > 0$ for $w_t h_t > \frac{\lambda\mu h_{t-1}}{\phi\beta\delta z} = \bar{w}$, therefore, $\bar{w}$ is a threshold effective wage rate at which an agent begins to invest in the education of children. The economic intuition underlying this threshold value can be derived by comparing the marginal benefit/marginal cost ratio of educating children with the marginal benefit/marginal cost ratio of raising a child. More precisely, the marginal benefit of an additional child is given by $\frac{dU_t}{dn_t} = \frac{\lambda}{n_t}$, while the marginal cost is given by $w_t h_t z$ at $e_t = 0$ and consequently the ratio of marginal benefit on marginal cost is: $\frac{\lambda}{n_t w_t h_t z}$. Regarding marginal benefit/marginal cost ratio of educating children, the marginal benefit is given by $\frac{dU_t}{dh_{t+1}} \frac{dh_{t+1}}{de_t} = \frac{\beta\delta}{\mu h_{t-1}}$ at $e_t = 0$. The marginal cost of educating children is $\frac{d(e_t n_t)}{de_t} = n_t$. Therefore the benefit/marginal cost ratio of educating children is equal to: $\frac{\beta\delta}{\mu h_{t-1} n_t}$. If the former ratio is smaller than the latter, i.e., $\gamma > 0$ or equivalently $w_t h_t > \bar{w}$, then parents prefer to invest on the education of their children rather than have another child. In the opposite case, in which the effective wage rate is lesser or equal to $\bar{w}$, the parents prefer to have more children rather than invest their wage on educating the children. As in De La Croix and Doepke (2003), the classical trade-off between the education of children (quality) and the number of children (quantity) emerges.

The function γ plays a crucial role in distinguishing between preferences among quality and quantity of children depending on the income level of a country. As long as the benefits associated with educating children (i.e. quality) are greater than the education costs, then $\gamma\left(w_t, h_t, h_{t-1}\right) > 0$, and the education level is set to a positive value, while the number of children results at a minimum level. The opposite occurs as long as $\gamma\left(w_t, h_t, h_{t-1}\right) \leq 0$ and the (private) education level becomes zero. The two situations are related to the effective wage rate level, and hence to the income of the country. While in high-income countries more importance is attached by parents to the education of children in comparison to their quantity, the opposite occurs in low-income countries, thus confirming empirical evidence previously described.

2.3. Firms

We assume that there are many competitive producers with the constant-returns-to-scale production technology. The capital stock is assumed to depreciate completely after one period of use.[3] Therefore, capital stock in a period is equal to savings in the previous one. Production in time t employs physical capital K_t and labor L_t. We denote by Q_t the aggregate output and represent the aggregate technology with the following Cobb–Douglas production function:

$$Q_t = F(K_t, L_t) = A K_t^{\alpha} L_t^{1-\alpha}, \tag{15}$$

where $A > 0$ is the total factor productivity and $\alpha \in (0, 1)$ is the productivity of physical capital. The firm chooses inputs by maximizing profits $Q_t - w_t L_t - r_t K_t$. The profit maximization conditions are given as:

$$w_t := \frac{A(1-\alpha)K_t^{\alpha}}{L_t{}^{\alpha}} \tag{16}$$

and

$$r_t := \frac{A\alpha K_t^{\alpha-1}}{L_t{}^{\alpha-1}}. \tag{17}$$

Following Hirazawa and Yakita (2017), the labor L_t is given by

$L_t = h_t N_t$.

Define $k_t = \frac{K_t}{N_t}$. In this light, formulae (16) and (17) become

$$w_t = \frac{A(1-\alpha)k_t^{\alpha}}{h_t^{\alpha}} \tag{18}$$

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[3] As usual in the OLG models, we assume that the depreciation rate of capital is equal to one. This assumption derives from the fact that, in the OLG model, one period (generation) represents 20 or 30 years and therefore it is realistic to assume that capital depreciates completely after one period (see e.g. De La Croix and Michel (2002)).

and

$$r_t = A\alpha \frac{k_t^{\alpha-1}}{h_t^{\alpha-1}}. \tag{19}$$

Moreover, since we assume $K_{t+1} = s_t N_t$, $N_{t+1} = n_t N_t$ we have that

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{s_t N_t}{n_t N_t} = \frac{s_t}{n_t}. \tag{20}$$

This formula can also be encountered in Hirazawa and Yakita (2017).

2.4. Market equilibrium

Taking into account Eqs. (20) and (5) and, by considering Proposition 2.1, the final dynamical system T describing the evolution of capital per capita and that of human capital results in the following Proposition (see Appendix B for more details).[4]

Proposition 2.2. *Let $\frac{\beta\delta}{\lambda} \in (0,1)$, $v_t = h_{t-1}$, and*

$$\gamma(k_t, h_t, v_t) = \frac{\beta\delta}{\lambda} zA(1-\alpha)k_t^{\alpha} h_t^{1-\alpha} - \frac{\mu}{\phi} v_t, \tag{21}$$

then

$$T := \begin{cases} k_{t+1} = \begin{cases} \frac{\rho}{\lambda-\beta\delta}\ \left(zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t\right) & \text{if } \gamma > 0 \\ \frac{\rho}{\beta\delta}\ \frac{\mu}{\phi}\ v_t & \text{if } \gamma = 0 \\ \frac{\rho}{\lambda}\ z\ A(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} & \text{if } \gamma < 0 \end{cases} \\ h_{t+1} = \begin{cases} \pi \left[\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)\left(zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t\right)\right]^{\delta} h_t^{1-\delta} & \text{if } \gamma > 0 \\ \pi\left(\mu v_t\right)^{\delta} h_t^{1-\delta} & \text{if } \gamma \le 0 \end{cases} \\ v_{t+1} = h_t \end{cases} . \tag{22}$$

System (22) describes the evolution of physical capital per capita k_t and human capital h_t over time. Then, starting from a given initial state, the correspondent sequences can be obtained. From the inspection of the equations describing the equilibrium dynamics, it can be noted that the system is three-dimensional and it cannot be reduced.

In order to describe in which measure the role of grandparents in educating grandchildren affects the long-run evolution of the system and the child Q–Q trade-off, we recall the equilibria condition for the number of children n_t and education e_t coming from the conditions on the constrained maximization problem.

Thus, the optimal number of children at any time t and the optimal education level can be obtained by substituting (16) into the conditions in Proposition 2.2 together with the equality $h_{t-1} = v_t$. Quality e_t and quantity n_t result as follows:

$$e_t = \begin{cases} \frac{\beta\delta}{\lambda-\beta\delta} zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\lambda}{\lambda-\beta\delta}\frac{\mu}{\phi}v_t & \text{if } \gamma > 0 \\ 0 & \text{if } \gamma \le 0, \end{cases} \tag{23}$$

and

$$n_t = \begin{cases} \frac{\lambda-\beta\delta}{1+\rho+\lambda} \frac{1}{z - \frac{\mu}{\phi}\frac{v_t}{A(1-\alpha)k_t^{\alpha}h_t^{1-\alpha}}} & \text{if } \gamma > 0 \\ \frac{\lambda}{1+\rho+\lambda}\frac{1}{z} & \text{if } \gamma \le 0. \end{cases} \tag{24}$$

In Proposition 2.3, we state a property on the value of n_t (the proof is in Appendix C).

Proposition 2.3. *The maximum value of n_t is $\frac{\lambda}{1+\rho+\lambda}\frac{1}{z}$.*

Our model confirms the classic Q–Q trade-off: for $w_t h_t > \bar{w}$, i.e., for $\gamma > 0$, parents find it convenient to educate their children and raise fewer of them; conversely when the wage level is low $w_t h_t \le \bar{w}$, i.e., $\gamma \le 0$, parents do not educate their children but find it optimal to generate them. In this case, however, unlike Yakita (2010), the education of children does not depend on a spillover effect (the ability of children to absorb the human capital of their parents) but on the time spent by grandparents on the education of grandchildren (μ). In the case where wages are high (i.e. $\gamma > 0$), as the amount of time grandparents spend with grandchildren increases (μ), the parental education – that is substituted – is reduced, but this frees up more resources that lead parents, ceteris paribus, to raise more children.

Following Yakita (2010) and Kitaura and Yakita (2010), from system T we obtain the following two-dimensional dynamic system S of the physical capital/human capital ratio, namely $x_t = k_t/h_t$, and of the human capital growth rate, denoted by $y_t = h_t/v_t$. The following Proposition holds (the proof is in Appendix D).

Proposition 2.4. *Let $x_t = k_t/h_t$, $y_t = h_t/v_t$, $\frac{\beta\delta}{\lambda} \in (0,1)$ and define*

$$g(x_t, y_t) = \frac{\beta\delta}{\lambda} zA(1-\alpha)x_t^{\alpha} - \frac{\mu}{\phi}\frac{1}{y_t}.$$

Then the system T in (22) is equivalent to the following system $S : \mathbb{R}^2_{++} \to \mathbb{R}^2_{++}$.

$$S := \begin{cases} x_{t+1} = \begin{cases} \frac{\rho}{\pi}\frac{1}{\phi^{\delta}}\frac{1}{\lambda-\beta\delta}\left(\frac{\lambda}{\beta\delta} - 1\right)^{\delta}\left[zA(1-\alpha)x_t^{\alpha} - \frac{\mu}{\phi}\frac{1}{y_t}\right]^{1-\delta} & \text{if } g > 0 \\ \frac{1}{\pi}\frac{\rho}{\beta\delta\phi}\mu^{1-\delta}\left(\frac{1}{y_t}\right)^{1-\delta} & \text{if } g = 0 \\ \frac{1}{\pi}\frac{\rho}{\lambda} zA(1-\alpha)\mu^{-\delta}x_t^{\alpha}y_t^{\delta} & \text{if } g < 0 \end{cases} \\ y_{t+1} = \begin{cases} \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta}\left[zA(1-\alpha)x_t^{\alpha} - \frac{\mu}{\phi}\frac{1}{y_t}\right]^{\delta} & \text{if } g > 0 \\ \pi\mu^{\delta}\left(\frac{1}{y_t}\right)^{\delta} & \text{if } g \le 0. \end{cases} \end{cases} \tag{25}$$

Moreover, we have:

$$\frac{e_t}{h_t} = \begin{cases} \frac{\beta\delta}{\lambda-\beta\delta} zA(1-\alpha)x_t^{\alpha} - \frac{\lambda}{\lambda-\beta\delta}\frac{\mu}{\phi}\frac{1}{y_t} & \text{if } g > 0 \\ 0 & \text{if } g \le 0. \end{cases}$$

Notice that function g defined in Proposition 2.4 has the same meaning of function γ defined in Proposition 2.2, as it is obtained from the latter after a change in the variables. More precisely, $\gamma > (\le)0$ iff $g > (\le)0$. Hence positive values of g are associated to high-income countries (i.e. the effective wage rate is sufficiently high) and parents invest in education, while the opposite occurs when $g \le 0$.

The fraction $\frac{e_t}{h_t}$ is the ratio of parental spending on education per child at time t divided by the human capital of the parents themselves. Higher levels of parental human capital h_t generate more income and thus a higher level, ceteris paribus, of spending on children's education.

We can also express n_t as follows:

$$n_t = \begin{cases} \frac{\lambda-\beta\delta}{1+\rho+\lambda}\frac{1}{z - \frac{\mu}{\phi}\frac{1}{A(1-\alpha)x_t^{\alpha}y_t}} & \text{if } g > 0 \\ \frac{\lambda}{1+\rho+\lambda}\frac{1}{z} & \text{if } g \le 0 \end{cases}$$

confirming the lower choice about the quantity characterizing high-income countries (i.e. $g > 0$) in comparison to the low-income ones (i.e. $g \le 0$).

In order to understand how quality and quantity move as x_t or y_t changes, the following partial derivatives (when $g > 0$) can be easily obtained and their sign can be determined:

$$\frac{\partial \frac{e_t}{h_t}}{\partial x_t} = \frac{\beta\delta}{\lambda-\beta\delta} zA(1-\alpha)\alpha x_t^{\alpha-1} > 0,$$

$$\frac{\partial n_t}{\partial x_t} = -\frac{\lambda-\beta\delta}{1+\rho+\lambda}\left(z - \frac{\mu}{\phi}\frac{1}{A(1-\alpha)x_t^{\alpha}y_t}\right)^{-2}\frac{\mu}{\phi}\frac{\alpha}{A(1-\alpha)y_t}x_t^{\alpha-1} < 0.$$

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[4] We need to change the variable from h_{t-1} to v_t to move to a discrete-time system of equations of first order.

Then, in terms of comparative static, we can obtain information about the influence an increase in the physical–human capital ratio (x_t) or in the human capital growth rate (y_t) may have on quality (measured by e_t/h_t) and quantity (i.e. n_t) in high-income countries (i.e. the effective wage rate is high enough and $g > 0$). If the ratio of physical–human capital (x_t) increases, the wage also increases ($w_t = A(1-\alpha)x_t^{\alpha}$). The increase in income leads, on the one hand, to more resources being devoted to educating children and, on the other hand, at the same time, increases the cost of raising children by inducing parents to raise fewer children but educate them more. In addition, we observe that

$$\frac{\partial \frac{e_t}{h_t}}{\partial y_t} = \frac{\lambda}{\lambda - \beta\delta} \frac{\mu}{\phi} \frac{1}{y_t^2} > 0,$$

$$\frac{\partial n_t}{\partial y_t} = -\frac{\lambda - \beta\delta}{1+\rho+\lambda} \left(z - \frac{\mu}{\phi} \frac{1}{A(1-\alpha)x_t^{\alpha} y_t} \right)^{-2} \frac{\mu}{\phi} \frac{1}{A(1-\alpha)x_t^{\alpha} y_t^2} < 0.$$

Hence, if the growth rate of human capital increases, this increases the cost of raising children by inducing parents to raise fewer children but educate them more. Hence, it is immediately observed that, as long as $g(x_t, y_t) > 0$ then at any time t the education/human capital ratio (representing a quality measure) is increasing w.r.t. the physical/human capital ratio and to the human capital growth rate while the opposite relationship occurs when considering the number of children (representing a quantity measure) thus confirming the Q–Q trade-off.

The discussion of the opposite case, i.e. the effective wage rate is low enough ($g \leq 0$), is trivial as both quality and quantity results are constant and represent, respectively, the lower bound and the upper bound for the two corresponding variables (e_t and n_t).

3. Qualitative and quantitative dynamics

3.1. Equilibria and stability

We first provide an analytical discussion for system S in order to verify the existence of equilibria and their properties, where equilibria are time-invariant solutions of the system S. With this aim in mind, we define $D_- = \{(x_t, y_t) \in \mathbb{R}^2_{++} : g(x_t, y_t) < 0\}$ and $D_+ = \{(x_t, y_t) \in \mathbb{R}^2_{++} : g(x_t, y_t) > 0\}$. We do not consider the border separating D_+ from D_- as it represents a particular combination between variables and parameters in which, as can be easily verified, it can exist a single equilibrium point only for a unique combination of parameters constellation. Notice that the steady state corresponds to a balanced growth path at which the physical–human capital ratio, i.e., k_t/h_t remains constant over time.

The curve $g(x_t, y_t) = 0$ separates the set $\mathbb{R}^2_{++}$ into two subsets: one is related to combinations between state variables associated to positive levels of g (high-income countries), namely D_+, while the opposite occurs as long as $(x_t, y_t) \in D_-$ (low-income countries). It is immediately obvious that the curve $g(x_t, y_t) = 0$ describes a function $y_t = \tilde{g}(x_t)$ defined for all $x_t > 0$ that is continuous and differentiable, convex, strictly decreasing such that $\lim_{x_t \to +\infty} \tilde{g}(x_t) = 0$ and $\lim_{x_t \to 0^+} \tilde{g}(x_t) = +\infty$. Such a curve is visible in Fig. 1 being the border between the red and the blue region.

As far as the steady states of the system and their stability are concerned, similar to what has been proposed by Hirazawa and Yakita (2017), we distinguish between the following two cases, namely Phase I and Phase II.

Phase I: equilibrium in D_-

We first focus on the subset of $\mathbb{R}^2_{++}$ associated to $g(x_t, y_t) < 0$. In this region, the system admits a unique equilibrium point that is locally asymptotically stable as stated in Proposition 3.1, whose proof is in Appendix E.

Proposition 3.1. *Consider the system S as defined in (25). Then, there exists $\bar{\mu} \in (0, 1)$ such that for all $\mu > \bar{\mu}$, S admits a unique equilibrium point $E^*_- = \left(x^*_-, y^*_-\right) \in D_-$, with*

$$x^*_- = \left[\pi^{-\frac{1}{1+\delta}} \mu^{-\frac{\delta}{1+\delta}} \frac{\rho}{\lambda} zA(1-\alpha)\right]^{\frac{1}{1-\alpha}}$$

and

$$y^*_- = \pi^{\frac{1}{1+\delta}} \mu^{\frac{\delta}{1+\delta}},$$

that is locally asymptotically stable.

Trivial computations of the partial derivative of the equilibrium value w.r.t. μ enable us to conclude that, if the time devoted by grandparents to rearing grandchildren is high enough, then the human capital growth rate increases with μ. As a consequence, at the equilibrium, the physical capital/human capital ratio decreases w.r.t. μ. Notice that since k_t does not depend on μ (see system (22)) while the human capital level at each period is positively affected by μ (as it emerges from system (22)), then, at the steady state, the economy is characterized by a decreasing value of x^*_- as long as the time grandparents spend in rearing children increases. Then, the equilibrium is given by a greater level of human capital w.r.t. physical capital. As it is clear, the human capital growth rate at the steady state shows higher values as μ increases.

Regarding the role of other exogenous variables on the equilibrium points, as it occurs in other contributions, it can be observed that the total productivity factor and the technology coefficient in the human capital production positively affect both the physical–human capital equilibrium ratio and the human capital equilibrium growth rate.

Phase II: equilibrium in D_+

We now consider combinations $(x_t, y_t) \in \mathbb{R}^2_{++}$ such that $g(x_t, y_t) > 0$. Unfortunately, in this phase, we are not able to analytically determine the possible equilibrium points of system S because we have to deal with a transcendental equation whose exact solution is not available. Nevertheless, some results related to the existence and stability of the steady state can be given. They are summarized in Proposition 3.2 that is proved in Appendix F.

Proposition 3.2. *System S defined in (25) admits at most two equilibrium points in D_+. Moreover, there exists $\bar{\bar{\mu}} \in (0, 1)$ such that for all $\mu < \bar{\bar{\mu}}$ the equilibrium point $E^*_+ \in D_+$ is unique and it is locally asymptotically stable.*

Anyway, as it has been proved in Appendix F, we can easily express y^* in terms of x^*. In particular, we find $y^* = \pi^{\frac{1}{1-\delta}} \left(\frac{\beta\delta\phi}{\rho}\right)^{\frac{\delta}{1-\delta}} (x^*)^{\frac{\delta}{1-\delta}}$, so that the equilibrium values of the two variables are positively correlated (i.e., they both increase or decrease as some parameters are moved). By taking into account the equation in the system (25), it is easy to observe that, as long as μ increases, then, the human capital growth rate from time t to time $t+1$ decreases, and the same occurs for the physical capital/human capital ratio. As a consequence, since both k_t and h_t decrease from one period to the following as μ increases (see system (22)), then, from the previous considerations it follows that as long as the time grandparents devote to educating grandchildren increases, the capital per capital reduction is stronger than the one of the human capital. As far as the present case is concerned, unfortunately, more analytical results cannot be added.

In Fig. 1, we show the results previously described from the qualitative point of view. The blue region is the set D_+ while the red one is the set D_-; the curve separating the two regions is described by the equation $g(x_t, y_t) = 0$. In panel (a), we fix μ at a low value, while in panel (b) μ is high enough. The point represented in both panels corresponds to the physical capital/human capital equilibrium ratio x^* and to the human capital equilibrium growth rate y^*, i.e., the locally stable interior steady state as described in Propositions 3.1 and 3.2.

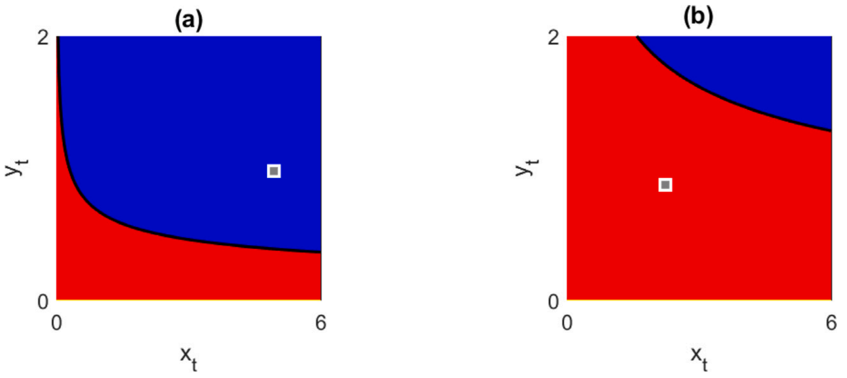

Fig. 1. The sets D_+ (in blue) and D_- (in red) and the locally stable fixed point for low μ (panel (a)) and high μ (panel (b)). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Parameter ranges with their most common values.

Parameter	Range	Particular values
α	$0 < \alpha < 1$	$\alpha = 0.25$ in Goldberger (1968) $\alpha = \frac{1}{3}$ in De La Croix and Doepke (2003) and Coppier et al. (2021)
A	$A > 0$	$A = 1.01$ in Goldberger (1968) $A = 60$ in Coppier et al. (2021)
β	$0 < \beta < 1$	$\beta = 0.1$ in Coppier et al. (2021) $\beta = 0.169$ in De la Croix and Doepke (2004)
δ	$0 < \delta < 1$	$\delta = 0.6$ in De la Croix and Doepke (2004) $\delta = 0.7$ in Coppier et al. (2021)
ϕ	$0 < \phi \leq 1$	$\phi = 1$ in Yakita (2010) and De la Croix and Doepke (2004)
λ	$0 < \lambda < 1$	$\lambda = 0.1$ in Hirazawa and Yakita (2009) $\lambda = 0.3$ in Coppier et al. (2021)
π	$\pi > 0$	$\pi = 1$ in Coppier et al. (2021) and Cipriani and Fioroni (2019)
ρ	$0 < \rho < 1$	$\rho = 0.1$ in Coppier et al. (2021) $\rho = 0.99^{120}$ in De La Croix and Doepke (2003)
z	$0 < z < 1$	$z = 0.075$ in De la Croix and Doepke (2004) $z = 0.1$ in Coppier et al. (2021)
θ	$0 < \theta < 1$	$\theta = 0.3$ in Hirazawa and Yakita (2017)

Given the analytical complexity of system S and the high number of parameters, some open questions cannot be approached analytically. For instance, if two coexisting equilibria may emerge in D_+, which is the structure of their basins? Or, in addition, if parameter combinations exist such that both E_+^* and E_-^* are owned by the system, again, which is the structure of their basins? Or, finally, which are the properties of equilibria, and how do they change with parameters? Such open questions may only find answers by using a numerical approach.

3.2. Numerical simulations

From the quantitative point of view, what is of interest is related to the evaluation of the iterated sequences in order to analyze the behavior of the human capital growth rate and of the physical/human capital ratio, providing that active aging can be a virtual transmission channel from old to young generations. Given the analytical complexity of the model, we will discuss such issues by means of numerical simulations.

In order to show the quantitative feature of the model, in Table 1 we recall the parameters involved in the model together with their range and, when available, the most common values used in literature.

Then, we set the parameters and we let μ vary in order to describe the role of transmission of human capital from the old age generation to the young one. By taking into account the suggested values as specified in Table 1, we consider the following parameter constellation:

$$\alpha = 1/3,\ \beta = 0.1,\ \lambda = 0.1,\ \delta = 0.6,\ \phi = 1,\ \pi = 1,\ \rho = 0.2994,\ z = 0.075,\ \theta = 0.3,\ \text{and}\ A = 10.$$

As far as the open questions are concerned, after several numerical experiments it turned out that:

(i) the fixed point $E_+^* \in D_+$ is unique providing that, as for the case $g(x_t, y_t) < 0$, also in the second phase no more than one equilibrium may exist;
(ii) the fixed points $E_+^* \in D_+$ and $E_-^* \in D_-$ do not coexist, as a consequence system S always admits a unique fixed point and, depending on the parameter constellation, it may belong to the region D_+ or D_-.

From the above-mentioned evidence, it follows that the unique fixed point E_j^*, $j \in \{+, -\}$, seems to be *globally* asymptotically stable, in the sense that it attracts all trajectories exiting from an initial condition in $D_+ \cup D_-$, thus supporting its relevance.

We start by showing the dynamic path of the physical capital/human capital ratio over time for a given initial condition (see Fig. 2) and two different μ values. As expected, both trajectories converge to the steady state equilibrium value that is given by E_+^* for low values of μ or by E_-^* for high values of μ.

In Figs. 3 and 4, we also present the phase diagrams of the same orbits depicted in Fig. 2, using the same parameter values and initial conditions. In particular, Fig. 3 shows the phase diagram for $\mu = 0.2$, and Fig. 4 for $\mu = 0.7$.

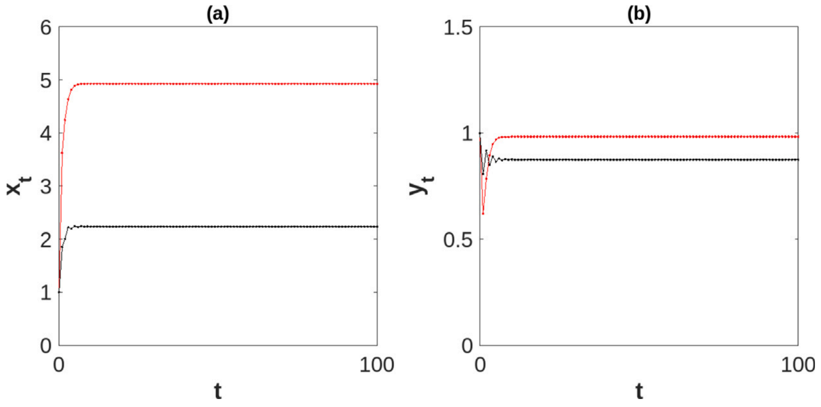

Fig. 2. Evolution over time for $x_0 = 1$ and $y_0 = 1$; for $\mu = 0.2$ (in red) and $\mu = 0.7$ (in black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

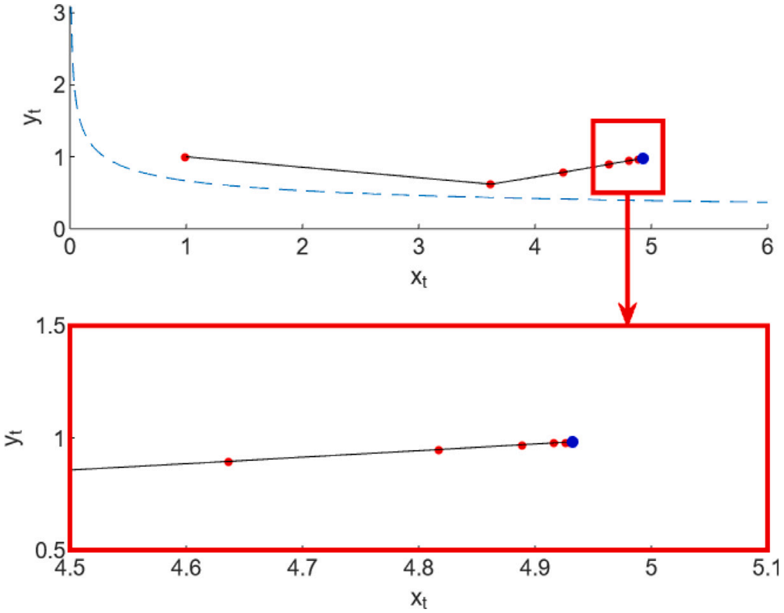

Fig. 3. Phase diagram for $x_0 = 1$, $y_0 = 1$, and $\mu = 0.2$.

The convergence to an equilibrium point can also be appreciated in Fig. 5 where the asymptotic values converging to the unique positive steady state for x_t and y_t are plotted for $\mu \in (0,1)$. The bifurcation diagrams show a modification at $\mu = \bar{\mu}$ as defined in Proposition 3.1. When μ crosses $\bar{\mu}$, the stable fixed point passes from D_+ to D_- crossing the border $g(x_t, y_t) = 0$ and represented in Fig. 1. Notice that the border crossing bifurcation does not change the stability property of the steady state, but it influences its modification w.r.t. μ. To be more precise, while x^* decreases as μ increases, the human capital equilibrium growth rate is not monotonic w.r.t. μ, passing from a decreasing branch (as long as $g(x_t, y_t) > 0$) to an increasing one (associated to $g(x_t, y_t) < 0$).

The results of Fig. 5 confirm the Phase I and Phase II comments. In fact, as the time that grandparents dedicate to educating their grandchildren increases, the physical–human capital ratio decreases in both phases. As regards the growth rate of human capital, however, in Phase II, i.e. when $w_t h_t > \bar{w}$, an increase in the time spent by grandparents educating their grandchildren reduces the rate of human capital growth as parents will use fewer resources to educate their children and will raise more. Differently, in Phase I, i.e. when $w_t h_t < \bar{w}$, parents have limited resources and therefore raise children without educating them. In this case, an increase in the time spent by grandparents educating their grandchildren increases the growth rate of human capital.

Regarding the effect of the time spent by grandparents in educating their grandchildren on capital accumulation, the main results are presented in Fig. 6 where the growth rate of physical capital and human capital is reported for two different levels of μ. Low (high) μ corresponds to lower (higher) capital accumulation levels, leading to lower (higher) growth rate for the human capital of the grandchildren. This is due to the fact that when grandparents spend more time with grandchildren then they allow resources to be freed up thus representing a channel for more growth.

Finally, in Fig. 7 we want to investigate the relationship between the quantity–quality of children. As far as the Q–Q trade-off is concerned, it is possible to observe that for high values of μ (i.e. $\mu > \bar{\mu}$ and the equilibrium point in D_-), quantity reaches the maximum value while the education/human capital ratio reaches the minimum value. In fact, in this case, parents have low income and therefore raise children without educating them. Hence, the possibility that grandparents will spend part of their time educating their grandchildren allows low-income countries to raise children and also to educate them (in the absence of resources spent by parents on education). On the other hand, for lower values of μ corresponding to the case in which the steady-state belongs to D_+, quantity and quality move in opposite directions as long as μ increases. In this case, the possibility that grandparents spend part of their time educating their grandchildren, frees up resources for

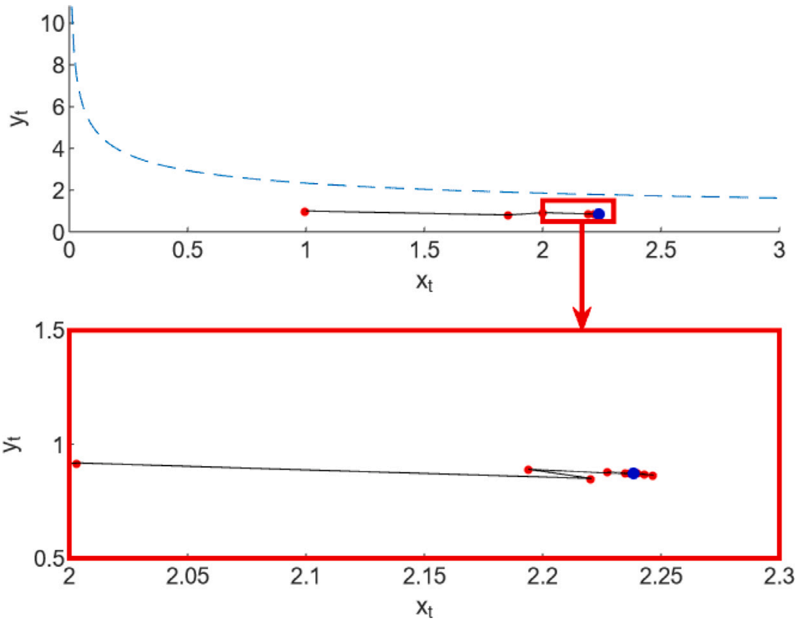

Fig. 4. Phase diagram for $x_0 = 1$, $y_0 = 1$, and $\mu = 0.7$.

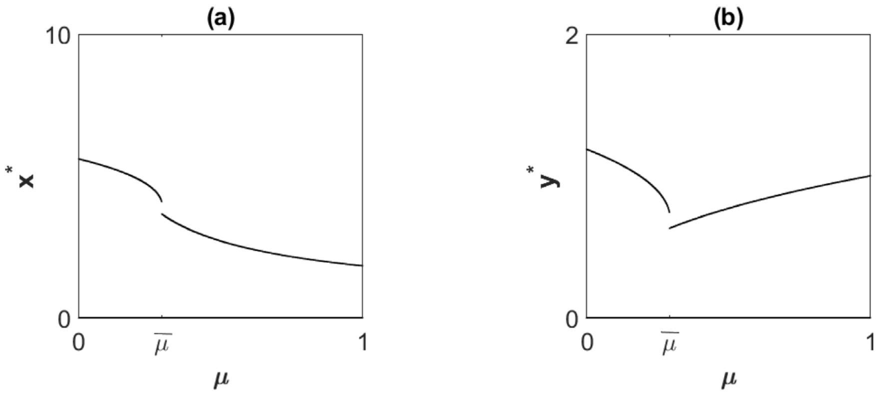

Fig. 5. One dimensional bifurcation diagrams w.r.t. μ.

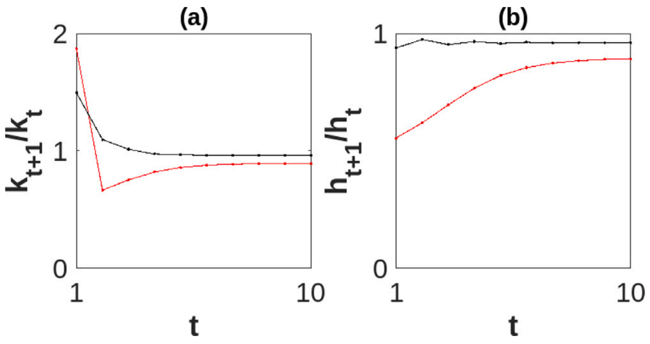

Fig. 6. Evolution over time of the physical capital index rate and of the human capital index rate for a given initial condition and tow values of μ: $\mu = 0.25$ (in red) and $\mu = 0.9$ (in black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

parents that can be used to raise more children (partially reducing the Q–Q trade-off).

4. Conclusions and further developments

We developed an overlapping generations model to study the potential role played by grandparents in the process of grandchildren's education and its effect on the evolution of human capital. We considered the quantity–quality trade-off faced by adults, i.e., parents' choice of how many children to have and their level of education. We have also incorporated an active role of grandparents who can contribute to the grandchildren's education by using part of their time. The resulting dynamic model has been studied by combining analytical tools and numerical simulations. Our main results highlight the important role that elderly people who decide to give up part of their free time to contribute to their grandchildren's education can play on human capital accumulation, fertility, and economic growth. Indeed, increasing life expectancy, especially but not only in high-income developed countries, coupled with increasing quality of life yields a potential role for active aging as a driving mechanism for growth and human capital accumulation, thus confirming the important role of the elderly in the transmission of human capital from the older generation to the younger generation.

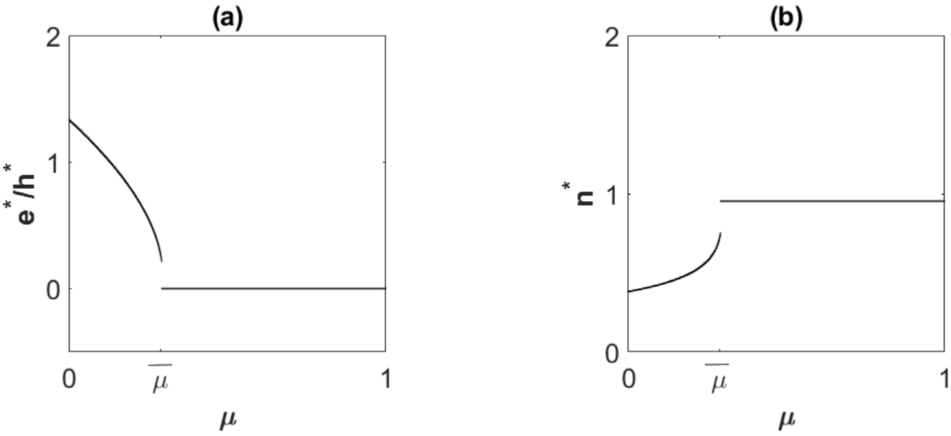

Fig. 7. One dimensional bifurcation diagrams w.r.t. μ.

On the one hand, the inclusion of time spent by grandparents to educate grandchildren allows for "mitigation" of the Q–Q trade-off. In low-income countries, parents only invest in their children's growth but fail to invest in their children's education: the presence of grandparents in the formation of their grandchildren's human capital allows, being the number of children constant, higher education level, thus, making the Q–Q trade-off less stringent. In high-income countries, education by parents can be partially reduced as it is replaced by the active role of grandparents. This frees up economic resources that allow parents to raise, ceteris paribus, more children, mitigating, again, the Q–Q trade-off. On the other hand, our model is able to show that when grandparents spend more time with grandchildren then they allow resources to be freed up thus representing a driving force for human capital accumulation and growth.

The present work represents a first step in the investigation of the role grandparents may play as a driver in human capital accumulation and economic growth in an OLG model. A natural development consists of moving from exogenous to endogenous choice, by adding such a component into the decision process (i.e. in the utility function and in the opportune constraints). The complexity of the model will result in increased and its traceability could be compromised. Anyway, we plan to try to explore such a study in the next step of our research line.

Another development may concern a possible endogenization longevity of grandparents, i.e. considering the introduction in the model of endogenous surviving probability (see e.g. Hirazawa and Yakita (2017)), a possible endogenization of the time spent by grandparents in educating grandchildren, and in considering that grandparents can contribute not only to the education of grandchildren but can also raise grandchildren. Another possible future development consists in endogenizing the child-rearing cost per child.

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Disclaimer

CRediT authorship contribution statement

Mauro Maria Baldi: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Raffaella Coppier:** Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Elisabetta Michetti:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to proof- read the article. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

Appendix A

In this appendix, we show how to derive first-order necessary conditions and second-order sufficient conditions to model (6)–(9).

(i) We reformulate the whole model substituting (5) into (2) and (4) into (3):

$$\begin{aligned} \max \quad U_t \quad &= \quad \ln c_{1,t} + \rho \left[\ln c_{2,t+1} + \theta \ln(1-\mu)\right] + \beta \ln \left[\pi \left(h_{t-1}\mu + \phi e_t\right)^{\delta} \bar{h}_t^{1-\delta}\right] \\ &\quad + \lambda \ln n_t \qquad \text{(A.1)} \\ \text{s. t.:} \quad & c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} + n_t(e_t + z w_t h_t) - w_t h_t = 0. \qquad \text{(A.2)} \end{aligned}$$

The gradient of the function expressing the constraint is always different to $\mathbf{0}$ and so the constraint qualification is always satisfied. The Lagrangian function of the maximization problem (A.1)–(A.2) is:

$$\begin{aligned} \mathcal{L} = &\ln c_{1,t} + \rho\{\ln c_{2,t+1} + \theta \ln(1-\mu)\} + \beta \ln \left[\pi \left(h_{t-1}\mu + \phi e_t\right)^{\delta} \bar{h}_t^{1-\delta}\right] \\ &+ \lambda \ln n_t - \eta_t \left[c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} + n_t(e_t + z w_t h_t) - w_t h_t\right], \qquad \text{(A.3)} \end{aligned}$$

where η_t is the Lagrangian multiplier associated to constraint (A.2). Setting the partial derivatives of $\mathcal{L}$ with respect to the decision variables $c_{1,t}$, $c_{2,t+1}$, n_t, and e_t, and to the Lagrangian multiplier η_t equal to zero, we find the following conditions:

$$\frac{\partial \mathcal{L}}{\partial c_{1,t}} = 0 \quad \Longrightarrow \quad \frac{1}{c_{1,t}} - \eta_t = 0, \qquad \text{(A.4)}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2,t+1}} = 0 \quad \Longrightarrow \quad \frac{\rho}{c_{2,t+1}} - \frac{\eta_t}{r_{t+1}} = 0, \tag{A.5}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \quad \Longrightarrow \quad \frac{\lambda}{n_t} - \eta_t \left(e_t + z w_t h_t\right) = 0, \tag{A.6}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial e_t} = 0 \quad \Longrightarrow \quad & \frac{\beta}{\pi \left(h_{t-1}\mu + \phi e_t\right)^{\delta} \bar{h}_t^{1-\delta}} \pi\delta \left(h_{t-1}\mu + \phi e_t\right)^{\delta-1} \bar{h}_t^{1-\delta}\phi \\ & - \eta_t n_t = 0, \end{aligned} \tag{A.7}$$

$$\frac{\partial \mathcal{L}}{\partial \eta_t} = 0 \quad \Longrightarrow \quad c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} + n_t(e_t + z w_t h_t) - w_t h_t = 0. \tag{A.8}$$

If we solve Eqs. (A.4), (A.5), (A.6), and (A.7) respectively by $c_{1,t}$, $c_{2,t+1}$, n_t, and e_t we get:

$$c_{1,t} = \frac{1}{\eta_t}, \tag{A.9}$$

$$c_{2,t+1} = \frac{\rho}{\eta_t} r_{t+1}, \tag{A.10}$$

$$n_t = \frac{1}{\eta_t} \frac{\lambda}{e_t + z w_t h_t}, \tag{A.11}$$

$$e_t = \frac{1}{\eta_t} \frac{\beta\delta}{n_t} - h_{t-1}\frac{\mu}{\phi}. \tag{A.12}$$

Plugging (A.9)–(A.11) into (A.2), after some manipulations we get:

$$\eta_t = \frac{1 + \rho + \lambda}{w_t h_t}. \tag{A.13}$$

Substituting such a value into (A.9)–(A.11), we find that

$$c_{1,t} = \frac{w_t h_t}{1 + \rho + \lambda}, \tag{A.14}$$

$$c_{2,t+1} = \frac{\rho}{1 + \rho + \lambda} r_{t+1} w_t h_t, \tag{A.15}$$

$$n_t = \frac{\lambda}{1 + \rho + \lambda} \frac{1}{z + \frac{e_t}{w_t h_t}}. \tag{A.16}$$

Moreover, from (8) and (A.15) we find that

$$s_t = \frac{\rho}{1 + \rho + \lambda} w_t h_t. \tag{A.17}$$

If we substitute (A.13) and (A.16) into (A.12), we obtain an equation with the only unknown e_t:

$$\left(1 - \frac{\beta\delta}{\lambda}\right) e_t = \frac{\beta\delta}{\lambda} z w_t h_t - \frac{\mu}{\phi} h_{t-1}, \tag{A.18}$$

where the right hand side is the already introduced γ. By hypothesis, $\frac{\beta\delta}{\lambda} \neq 1$ and so (A.18) is never indeterminate. Depending on the values of the left and right-hand side, e_t could be, in principle, either positive or negative. Since negative values of e_t are not acceptable, we set $e_t = 0$ when a particular combination of values w_t, h_t, and h_{t-1} would make $e_t < 0$ in (A.18). With this premise and with appropriate final substitutions in the presented equations, the stated first order conditions can be easily found.

Concerning the second-order condition, we consider the reformulation (A.1)–(A.2) of the maximization problem with its Lagrangian function (A.3). Let $g(c_{1,t}, c_{2,t+1}, n_t, e_t) = c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} + n_t(e_t + z w_t h_t) - w_t h_t$ be the function associated to the constraint (A.2). We study the bordered Hessian matrix $\mathbf{H}$ given by

$$\begin{aligned} \mathbf{H} &= \begin{bmatrix} 0 & g'_{c_{1,t}} & g'_{c_{2,t+1}} & g'_{n_t} & g'_{e_t} \\ g'_{c_{1,t}} & \mathcal{L}''_{c_{1,t},c_{1,t}} & \mathcal{L}''_{c_{1,t},c_{2,t+1}} & \mathcal{L}''_{c_{1,t},n_t} & \mathcal{L}''_{c_{1,t},e_t} \\ g'_{c_{2,t+1}} & \mathcal{L}''_{c_{2,t+1},c_{1,t}} & \mathcal{L}''_{c_{2,t+1},c_{2,t+1}} & \mathcal{L}''_{c_{2,t+1},n_t} & \mathcal{L}''_{c_{2,t+1},e_t} \\ g'_{n_t} & \mathcal{L}''_{n_t,c_{1,t}} & \mathcal{L}''_{n_t,c_{2,t+1}} & \mathcal{L}''_{n_t,n_t} & \mathcal{L}''_{n_t,e_t} \\ g'_{e_t} & \mathcal{L}''_{e_t,c_{1,t}} & \mathcal{L}''_{e_t,c_{2,t+1}} & \mathcal{L}''_{e_t,n_t} & \mathcal{L}''_{e_t,e_t} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & \frac{1}{r_{t+1}} & e_t & n_t \\ 1 & -\frac{1}{c_{1,t}^2} & 0 & 0 & 0 \\ \frac{1}{r_{t+1}} & 0 & -\frac{\rho}{c2,t+1^2} & 0 & 0 \\ e_t & 0 & 0 & -\frac{\lambda}{n_t^2} & -\eta_t \\ n_t & 0 & 0 & -\eta_t & -\frac{\beta\delta\phi^2}{(h_{t-1}\mu+\phi e_t)^2} \end{bmatrix}. \end{aligned} \tag{A.19}$$

According to constrained optimization theory, if we show that the last three leading principal minors M_3, M_4, and M_5 (respectively of order 3, 4, and 5) of $\mathbf{H}$ alternate in sign such that $M_3 > 0$, $M_4 < 0$, and $M_5 > 0$, then conditions (10)–(14) are also sufficient. We have:

$$M_3 = \frac{1}{r_{t+1}^2 c_{1,t}^2} + \frac{\rho}{c_{2,t+1}^2} > 0,$$

$$M_4 = -\frac{\lambda}{r_{t+1}^2 c_{1,t}^2 n_t^2} - \frac{\rho\lambda}{c_{2,t+1}^2 n_t^2} - \frac{\rho e_t^2}{c_{2,t+1}^2 c_{1,t}^2} < 0,$$

and

$$\begin{aligned} M_5 = {} & \left(\frac{\rho}{c_{2,t+1}^2} + \frac{1}{c_{1,t}^2 r_{t+1}^2}\right)\left(\frac{\lambda}{n_t^2} \frac{\beta\delta\phi^2}{\left(h_{t-1}\mu + \phi e_t\right)^2} - \eta_t^2\right) \\ & + \frac{\rho}{c_{1,t}^2 c_{2,t+1}^2}\left(\frac{\beta\delta\phi^2 e_t^2}{\left(h_{t-1}\mu + \phi e_t\right)^2} - 2 n_t e_t \eta_t + \lambda\right). \end{aligned} \tag{A.20}$$

M_5 is a cumbersome expression and, in principle, is not necessarily positive. For these reasons, we proceed in multiple steps. First, we plug into (A.20) the values of $c_{1,t}$ and $c_{2,t+1}$ from the first-order conditions. With these substitutions, M_5 reduces to:

$$\begin{aligned} M_5 = {} & \frac{(1+\rho+\lambda)^2}{\rho r_{t+1}^2 w_t^2 h_t^2} \left[(1+\rho)\left(\frac{\lambda}{n_t^2} \frac{\beta\delta\phi^2}{\left(h_{t-1}\mu + \phi e_t\right)^2} - \eta_t^2\right)\right. \\ & \left. + \frac{(1+\rho+\lambda)^2}{w_t^2 h_t^2}\left(\frac{\beta\delta\phi^2 e_t^2}{\left(h_{t-1}\mu + \phi e_t\right)^2} - 2 n_t e_t \eta_t + \lambda\right)\right]. \end{aligned} \tag{A.21}$$

The next step consists in replacing in (A.21) the values of n_t, e_t, and η_t coming from the first-order conditions. Note that we can express e_t and n_t in a more compact form. In fact, $e_t = \frac{\lambda}{\lambda-\beta\delta}\gamma$. If we define $\chi = z w_t h_t - \frac{\mu}{\phi} h_{t-1}$, then $n_t = \frac{\lambda-\beta\delta}{1+\rho+\lambda} \frac{w_t h_t}{\chi}$. It is also convenient to express the term $h_{t-1}\mu + \phi e_t$ as $\frac{\beta\delta\phi}{\lambda-\beta\delta}\chi$. With these substitutions, (A.21) becomes:

$$M_5 = \frac{(1+\rho+\lambda)^4}{\rho r_{t+1}^2 w_t^4 h_t^4} \left[(1+\rho)\left(\frac{\lambda}{\beta\delta} - 1\right) + \lambda\left(\frac{\lambda}{\beta\delta}\frac{\gamma^2}{\chi^2} - 2\frac{\gamma}{\chi} + 1\right)\right]. \tag{A.22}$$

By hypothesis, $\frac{\beta\delta}{\lambda} \in (0, 1)$ and so $\frac{\lambda}{\beta\delta} > 1$. Consequently, the first addend within the square brackets in (A.22) is positive. We also have that:

$$\frac{\lambda}{\beta\delta}\frac{\gamma^2}{\chi^2} - 2\frac{\gamma}{\chi} + 1 > \frac{\gamma^2}{\chi^2} - 2\frac{\gamma}{\chi} + 1 = \left(\frac{\gamma}{\chi} - 1\right)^2 \geq 0.$$

Thus, the second addend is never negative and so M_5 is always positive. □

Appendix B

We introduce a dummy variable v_t such that $v_{t+1} = h_t$. Substituting (18) into the expression of γ introduced in Proposition 2.1 and considering that $h_{t-1} = v_t$, it is straightforward to derive the new expression of γ.

To derive the expression of k_{t+1} when $\gamma > 0$, we consider the equation $k_{t+1} = \frac{s_t}{n_t}$ and we substitute the values of s_t and n_t respectively given by (12) and (14) when $\gamma > 0$. We have:

$$k_{t+1} = \frac{s_t}{n_t} = \frac{\rho}{\lambda - \beta\delta}\left(z w_t h_t - \frac{\mu}{\phi} h_{t-1}\right). \tag{B.1}$$

Then, using (18) and considering that $h_{t-1} = v_t$, (B.1) becomes:

$$k_{t+1} = \frac{\rho}{\lambda - \beta\delta}\left(z A(1-\alpha) k_t^{\alpha} h_t^{1-\alpha} - \frac{\mu}{\phi} v_t\right).$$

The cases when $\gamma = 0$ and $\gamma < 0$ are carried out in the same way.

The equations related to h_{t+1} can be found following analogous steps, this time starting from (5) and assuming that $\bar{h}_t = h_t$. □

Appendix C

According to (24), the value $\frac{\lambda}{1+\rho+\lambda}\frac{1}{z}$ is already reached when $\gamma \leq 0$. To prove that this value is also maximum, consider the case when $\gamma > 0$. Then, the following chain of inequalities holds:

$$\frac{\beta\delta}{\lambda} zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} > \frac{\mu}{\phi}v_t$$

$$z - \frac{\mu}{\phi}\frac{v_t}{A(1-\alpha)k_t^{\alpha}h_t^{1-\alpha}} > z - \frac{\beta\delta}{\lambda}z = z\frac{\lambda-\beta\delta}{\lambda}$$

$$n_t = \frac{\lambda-\beta\delta}{1+\rho+\lambda}\frac{1}{z - \frac{\mu}{\phi}\frac{v_t}{A(1-\alpha)k_t^{\alpha}h_t^{1-\alpha}}} < \frac{\lambda}{1+\rho+\lambda}\frac{1}{z}. \quad \square$$

Appendix D

Consider (21) and let $h_t > 0$. Then,

$$g = \frac{\gamma}{h_t} = \frac{\beta\delta}{\lambda} zA(1-\alpha)k_t^{\alpha}h_t^{-\alpha} - \frac{\mu}{\phi}\frac{v_t}{h_t} = \frac{\beta\delta}{\lambda} zA(1-\alpha)x_t^{\alpha} - \frac{\mu}{\phi}\frac{1}{y_t}. \tag{D.1}$$

Since $h_t \geq 0$, it follows that $g \geq 0$ iff $\gamma \geq 0$. When $\gamma > 0$, from system T we have that

$$\begin{aligned} x_{t+1} = \frac{k_{t+1}}{h_{t+1}} &= \frac{\frac{\rho}{\lambda-\beta\delta}\left(zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t\right)}{\pi\left[\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)\left(zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t\right)\right]^{\delta} h_t^{1-\delta}} \\ &= \frac{\rho}{\pi}\frac{1}{\phi^{\delta}}\frac{1}{\lambda-\beta\delta}\left(\frac{\lambda}{\beta\delta}-1\right)^{\delta}\left[\frac{zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t}{h_t}\right]^{1-\delta} \\ &= \frac{\rho}{\pi}\frac{1}{\phi^{\delta}}\frac{1}{\lambda-\beta\delta}\left(\frac{\lambda}{\beta\delta}-1\right)^{\delta}\left[zA(1-\alpha)x_t^{\alpha} - \frac{\mu}{\phi}\frac{1}{y_t}\right]^{1-\delta}. \end{aligned}$$

Likewise,

$$\begin{aligned} y_{t+1} = \frac{h_{t+1}}{v_{t+1}} = \frac{h_{t+1}}{h_t} &= \frac{\pi\left[\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)\left(zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t\right)\right]^{\delta} h_t^{1-\delta}}{h_t} \\ &= \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta}\left[\frac{zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\mu}{\phi}v_t}{h_t}\right]^{\delta} \\ &= \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta}\left[zA(1-\alpha)x_t^{\alpha} - \frac{\mu}{\phi}\frac{1}{y_t}\right]^{\delta}. \end{aligned}$$

The cases for $g = 0$ and $g < 0$ are carried out in an analogous way.

Concerning the ratio e_t/h_t, when $g > 0$ from (23) we have:

$$\begin{aligned} e_t &= \frac{\beta\delta}{\lambda-\beta\delta} zA(1-\alpha)k_t^{\alpha}h_t^{1-\alpha} - \frac{\lambda}{\lambda-\beta\delta}\frac{\mu}{\phi}v_t \\ &= h_t\left[\frac{\beta\delta}{\lambda-\beta\delta} zA(1-\alpha)\left(\frac{k_t}{h_t}\right)^{\alpha} - \frac{\lambda}{\lambda-\beta\delta}\frac{\mu}{\phi}\frac{v_t}{h_t}\right]. \end{aligned}$$

Consequently,

$$\frac{e_t}{h_t} = \frac{\beta\delta}{\lambda-\beta\delta} zA(1-\alpha)x_t^{\alpha} - \frac{\lambda}{\lambda-\beta\delta}\frac{\mu}{\phi}y_t.$$

Finally, from (23) we have that $e_t = 0$ when $g \leq 0$. Thus, $e_t/h_t = 0$ when $g \leq 0$. $\square$

Appendix E

Consider the dynamic system S when $g < 0$. Then, the equilibrium points satisfy the system of equations:

$$\begin{cases} x^* = \frac{1}{\pi}\frac{\rho}{\lambda} zA(1-\alpha)\mu^{-\delta}\left(x^*\right)^{\alpha}\left(y^*\right)^{\delta} \\ y^* = \pi\mu^{\delta}\left(\frac{1}{y^*}\right)^{\delta} \end{cases}$$

Since $y^* \neq 0$, we can multiply each side of the second equation of the system by $(y^*)^{\delta}$ and get the equivalent equation $(y^*)^{1+\delta} = \pi\mu^{\delta}$, whose solution is $y^* = \pi^{\frac{1}{1+\delta}}\mu^{\frac{\delta}{1+\delta}}$. Plugging such a value into the first equation of the system, this becomes:

$$x^* = \frac{\rho}{\lambda} zA(1-\alpha)\left(x^*\right)^{\alpha}\pi^{-\frac{1}{1+\delta}}\mu^{-\frac{\delta}{1+\delta}},$$

whose non-zero root is

$$x^* = \left[\frac{\rho}{\lambda} zA(1-\alpha)\pi^{\frac{2\delta-1}{1-\delta}}\mu^{\frac{2\delta^2-\delta}{1-\delta}}\right]^{\frac{1}{1-\alpha}}.$$

If we substitute the values of x^* and y^* into the expression of $g < 0$, we find an inequality that is solved for $\mu > \bar{\mu}$, where

$$\bar{\mu} = (\beta\delta\phi)^{\frac{(1+\delta)(1-\alpha)}{\alpha\delta+1-\alpha}}\left[\frac{zA(1-\alpha)}{\lambda}\right]^{\frac{1+\delta}{\alpha\delta+1-\alpha}}\rho^{\frac{\alpha(1+\delta)}{\alpha\delta+1-\alpha}}\pi^{\frac{1-2\alpha}{\alpha\delta+1-\alpha}}.$$

To prove that the point E^*_- is locally asymptotically stable, we first compute the Jacobian matrix $\mathbf{J}_-$ of the dynamic system S when $g < 0$. Let $f_1(x_t, y_t) = \frac{1}{\pi}\frac{\rho}{\lambda} zA(1-\alpha)\mu^{-\delta}x_t^{\alpha}y_t^{\delta}$ and $f_2(x_t, y_t) = \pi\mu^{\delta}y_t^{-\delta}$ be the two functions making up the map of system S when $g < 0$. Since $\frac{\partial f_2}{\partial x_t} = 0$, the Jacobian matrix will be in the form

$$\mathbf{J}_- = \begin{bmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{bmatrix}.$$

Since $\mathbf{J}_-$ is an upper triangular matrix we have:

$$J_{11} = \frac{\partial f_1}{\partial x_t} = \frac{1}{\pi}\frac{\rho}{\lambda} zA(1-\alpha)\mu^{-\delta}\alpha x_t^{\alpha-1}y_t^{\delta}, \tag{E.1}$$

and

$$J_{22} = \frac{\partial f_2}{\partial y_t} = -\pi\delta\mu^{\delta}y_t^{-\delta-1}. \tag{E.2}$$

If we plug the coordinates of E^*_- into (E.1) and (E.2), we find that $J_{11}(x^*, y^*) = \alpha$ and $J_{22}(x^*, y^*) = -\delta$. Since the modulus of each eigenvalue is less than one, the point E^*_- is locally asymptotically stable. $\square$

Appendix F

Consider the dynamic system S when $g > 0$. Then, the equilibrium points satisfy the system of equations:

$$\begin{cases} x^* = \frac{\rho}{\pi(\beta\delta\phi)^{\delta}(\lambda-\beta\delta)^{1-\delta}}\left[zA(1-\alpha)\left(x^*\right)^{\alpha} - \frac{\mu}{\phi}\frac{1}{y^*}\right]^{1-\delta} & \text{(a)} \\ y^* = \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta}\left[zA(1-\alpha)\left(x^*\right) - \frac{\mu}{\phi}\frac{1}{y^*}\right]^{\delta}. & \text{(b)} \end{cases} \tag{F.1}$$

Multiplying each side of (F.1a) by the corresponding side of (F.1b), we get the following equation:

$$x^*y^* = \frac{\rho}{\lambda-\beta\delta}\left[zA(1-\alpha)\left(x^*\right)^{\alpha} - \frac{\mu}{\phi}\frac{1}{y^*}\right].$$

This implies $zA(1-\alpha)(x^*)^{\alpha} - \frac{\mu}{\phi}\frac{1}{y^*} = \frac{\lambda-\beta\delta}{\rho}x^*y^*$. Substituting such a value into (F.1b), we get an equation where we can easily express y^* in terms of x^*. In particular, we find $y^* = \pi^{\frac{1}{1-\delta}}\left(\frac{\beta\delta\phi}{\rho}\right)^{\frac{\delta}{1-\delta}}(x^*)^{\frac{\delta}{1-\delta}}$. Substituting y^* into (F.1a), after some simple manipulations we find the following transcendental equation:

$$\begin{aligned} &\pi^{\frac{1}{1-\delta}}\left(\frac{\beta\delta\phi}{\rho}\right)^{\frac{\delta}{1-\delta}}\left(x^*\right)^{\frac{1}{1-\delta}} \\ &\quad = \frac{\rho}{\lambda-\beta\delta}\left[zA(1-\alpha)\left(x^*\right)^{\alpha} - \frac{\mu}{\phi}\pi^{-\frac{1}{1-\delta}}\left(\frac{\beta\delta\phi}{\rho}\right)^{-\frac{\delta}{1-\delta}}\left(x^*\right)^{-\frac{\delta}{1-\delta}}\right] \end{aligned} \tag{F.2}$$

Unfortunately, it is not possible to solve (F.2) analytically. Let $f(x)$ be the function associated to (F.2), i.e.:

$$\begin{aligned} f(x) = \pi^{\frac{1}{1-\delta}}\left(\frac{\beta\delta\phi}{\rho}\right)^{\frac{\delta}{1-\delta}}x^{\frac{1}{1-\delta}} - \frac{\rho}{\lambda-\beta\delta} \\ \times\left[zA(1-\alpha)x^{\alpha} - \frac{\mu}{\phi}\pi^{-\frac{1}{1-\delta}}\left(\frac{\beta\delta\phi}{\rho}\right)^{-\frac{\delta}{1-\delta}}x^{-\frac{\delta}{1-\delta}}\right]. \end{aligned}$$

For the sake of brevity, let us express $f(x)$ in the form $f(x) = ax^{\frac{1}{1-\delta}} - bx^{\alpha} + cx^{-\frac{\delta}{1-\delta}}$, where a, b, and c are appropriate positive constants depending on the parameters. We have:

$$f''(x) = \frac{a\delta}{(1-\delta)^2} x^{\frac{2\delta-1}{1-\delta}} + b\alpha(1-\alpha)x^{\alpha-2} + \frac{c\delta}{(1-\delta)^2 x^{\frac{-2+\delta}{1-\delta}}}.$$

Since $\alpha,\ \delta \in (0,\ 1)$, then $f''(x) > 0$ and so $f(x)$ is convex. Moreover, it is straightforward to see that $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to +\infty} f(x) = +\infty$. For all these reasons, $f(x)$ encounters the x axis in at most two points. This means that (F.2) has at most two roots and so the system S has at most two equilibrium points when $g > 0$.

Concerning the last part of the proof, let S_0 be the system S in the particular case when $g > 0$ and $\mu = 0$:

$$S_0 := \begin{cases} x_{t+1} = \frac{\rho}{\pi}\frac{1}{\phi^{\delta}}\frac{1}{(\lambda-\beta\delta)^{1-\delta}} \left[zA(1-\alpha)x_t^{\alpha}\right]^{1-\delta} \\ y_{t+1} = \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta} \left[zA(1-\alpha)x_t^{\alpha}\right]^{\delta} \end{cases}$$

The equilibrium points of S_0 are found solving the following system of equations:

$$\begin{cases} x^* = \frac{\rho}{\pi}\frac{1}{\phi^{\delta}}\frac{1}{(\lambda-\beta\delta)^{1-\delta}} \left[zA(1-\alpha)\,(x^*)^{\alpha}\right]^{1-\delta} \\ y^* = \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta} \left[zA(1-\alpha)\,(x^*)^{\alpha}\right]^{\delta} \end{cases}$$

A trivial solution to this system is $x^* = 0$ and $y^* = 0$. However, this solution is not acceptable because is on the border of the set D_+. With some manipulations, it is also possible to find the following solution:

$$x^* = \left\{ \frac{\rho}{\pi\,(\beta\delta\phi)^{\delta}} \left[\frac{zA(1-\alpha)}{\lambda-\beta\delta}\right]^{1-\delta} \right\}^{\frac{1}{1-\alpha+\alpha\delta}}, \tag{F.3}$$

and

$$y^* = \left\{ \pi^{1-\alpha}\,(\beta\delta\phi)^{\delta(1-\alpha)}\,\rho^{\alpha\delta} \left[\frac{zA(1-\alpha)}{\lambda-\beta\delta}\right]^{\delta} \right\}^{\frac{1}{1-\alpha+\alpha\delta}}. \tag{F.4}$$

When $\mu = 0$, $g(x_t, y_t) = \frac{\beta\delta}{\lambda} zA(1-\alpha)x_t^{\alpha}$. Thus, $g\,(x^*,\ y^*) > 0$ and so $(x^*,\ y^*)$ is a feasible equilibrium point to the system S_0. Let $f_1(x_t,\ y_t) = \frac{\rho}{\pi}\frac{1}{\phi^{\delta}}\frac{1}{(\lambda-\beta\delta)^{1-\delta}} \left[zA(1-\alpha)x_t^{\alpha}\right]^{1-\delta}$ and $f_2(x_t,\ y_t) = \pi\left(\frac{\beta\delta}{\lambda-\beta\delta}\phi\right)^{\delta} \left[zA(1-\alpha)\,x_t^{\alpha}\right]^{\delta}$ be the two functions making up the map of system S_0. Since $\frac{\partial f_1}{\partial y_t} = 0$ and $\frac{\partial f_2}{\partial y_t} = 0$, the Jacobian matrix will be in the form

$$\mathbf{J}_+ = \begin{bmatrix} J_{11} & 0 \\ J_{21} & 0 \end{bmatrix}.$$

Since $\mathbf{J}_+$ is a lower triangular matrix, we are only interested to know the value of J_{11} because it is one eigenvalue of $\mathbf{J}_+$ together with 0. We have:

$$J_{11} = \frac{\rho[zA(1-\alpha)]^{1-\delta}}{\pi\,(\beta\delta\phi)^{\delta}\,(\lambda-\beta\delta)^{1-\delta}}\frac{1}{x_t^{1-\alpha+\alpha\delta}} \tag{F.5}$$

Substituting the values of x^* and y^* into (F.5), we find:

$$J_{11}\left(x^*,\ y^*\right) = \alpha(1-\delta).$$

Since α and δ lie between 0 and 1, then $\left|J_{11}\right| < 1$. Being the other eigenvalue zero, we can conclude that $(x^*,\ y^*)$ is a locally stable equilibrium point. We point out that S tends to S_0 when $\mu \to 0$. Thus, there exists a $\bar{\bar{\mu}}$ such that the system S has only one equilibrium point and this equilibrium point is also locally stable. ☐

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